Spring 5-17-2010

Factorizations in the Irreducible Characters of Compact Semisimple Lie Groups

Andrew Rupinski
University of Pennsylvania, rupinski@math.upenn.edu

Factorizations in the Irreducible Characters of Compact Semisimple Lie Groups

Abstract

Our goal is to describe factorizations of the characters of irreducible representations of compact semisimple Lie groups. It is well-known that for a given Lie group G of rank n, the Virtual Representation Ring R(G) with the operations of tensor product, direct sum, and direct difference is isomorphic to a polynomial ring with integer coefficients and number of generators equal to n. As such, R(G) is a Unique Factorization Domain and thus, viewing a given representation of G as an element of this ring, it makes sense to ask questions about how a representation factors. Using various approaches we show that the types of factorizations which appear in the irreducible characters of G depend on the geometry of the root system and also have connections to the classifying space BG.

Degree Type

Dissertation

Degree Name

Doctor of Philosophy (PhD)

Graduate Group

Mathematics

First Advisor

Alexandre Kirillov

Keywords

Lie Groups, Irreducible Representations, Characters, Factorization

Subject Categories

Harmonic Analysis and Representation

FACTORIZATIONS IN THE IRREDUCIBLE CHARACTERS OF COMPACT SEMISIMPLE LIE GROUPS

Andrew Rupinski

A Dissertation

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2010

Alexandre Kirillov
Supervisor of Dissertation

Tony Pantev
Graduate Group Chairperson

Acknowledgments

Many people deserve thanks for getting me where I am. Ordered alphabetically by importance in Sanskirit, they include:

- My family without whom I would not be here, and without whose daily phone calls to check in on me I was able to get lots of work done.
- Janet, Monica, Robin and Paula in the office for always being there to tell me how to use the fax machine or to have me change the water as practice for my eventual career after grad school.
- My Advisor Dr. Kirillov, without whose help this dissertation would probably be complete gibberish instead of something that resembles mathematics. Also without his help, I probably would have had much longer proofs and several incorrect statements.
- The other professors in the department, especially my dissertation committee and in particular Dr. Chai's detailed comments on the final draft.
- All the other math grad students, especially Jason D. who probably sat through more discussions of my results than my advisor did.
- The janitorial staff in the department, because they probably don't get acknowledged very often for all that they do for us. Many mornings they were the only people here when I got into my office.
- The various non-math grad students I knew in any way for giving the illusion of knowing people outside the department.
- Everyone I know around Philadelphia through the Boy Scouts; they gave me plenty of stuff to do at night and on weekends so I wasn't always focused on math (even though I sometimes should have been).
- The basement of the library for giving this dissertation a good dark low-humidity home for the next thousand years. I thought about building a pyramid for its final storage place, but decided on the basement option because it was (1) required by the University and (2) cheaper.

ABSTRACT

FACTORIZATIONS IN THE IRREDUCIBLE CHARACTERS OF COMPACT SEMISIMPLE LIE GROUPS

Andrew Rupinski

Alexandre Kirillov, Advisor

Our goal is to describe factorizations of the characters of irreducible representations of compact semisimple Lie groups. It is well-known that for a given Lie group G of rank n, the Virtual Representation Ring $\mathfrak{R}(G)$ with the operations of $\otimes$, $\oplus$, and $\ominus$ is isomorphic to a polynomial ring with integer coefficients and number of generators equal to n. As such, $\mathfrak{R}(G)$ is a Unique Factorization Domain and thus, viewing a given representation of G as an element of this ring, it makes sense to ask questions about how a representation factors. Using various approaches we show that the types of factorizations which appear in the irreducible characters of G depend on the geometry of the root system and also have connections to the classifying space BG.

Contents

Chapter 1

Introduction

In this dissertation we seek to examine series of factorizations arising in the irreducible characters of a compact semisimple Lie group G. The operations of direct sum $\oplus$ and tensor product $\otimes$ make the set of finite-dimensional representations of G into a semiring; by introducing a formal operation $\ominus$, one may complete this semiring to a ring, the Virtual Representation Ring which we denote $\mathfrak{R}(G)$. With the operations of $\otimes$, $\oplus$, and $\ominus$, $\mathfrak{R}(G)$ is isomorphic to a polynomial ring with integer coefficients and number of generators equal to the rank of G; we are interested in classifying factorizations of irreducible characters when viewed as elements of $\mathfrak{R}(G)$. Formally all factorizations of representations are only factorizations of characters since the factors which arise in general are not characters of any finite dimensional G-module.

It is interesting that the question of factorization just among the irreducible representations turns out to be quite nontrivial, leading to a number of types of factorizations depending on the Lie group in question. For example, the irreducible characters of any compact semisimple Lie group possess two series of factorizations which we term ordinary factorizations (§3.1.1) and Γ-factorizations (§3.2.1). In addition, when G is a nonsimply-laced group, we show that there are three additional series: exotic factorizations (§3.1.2), ^{LS}-factorizations (§3.2.2), and Γ^{LS}-factorizations (§3.2.3).

In order to effectively deal with the irreducible representations, we usually choose to view them not in $\mathfrak{R}(G)$, but in a ring extension of $\mathfrak{R}(G)$ which we denote $\mathfrak{E}(G)$; this ring is the Virtual Character Ring of a maximal torus $\mathbb{T}(G) \subset G$. The Weyl group $W(G)$ acts on $\mathbb{T}(G)$, and hence acts on $\mathfrak{E}(G)$; it is well-known that the image of the embedding $\mathfrak{R}(G) \hookrightarrow \mathfrak{E}(G)$ is exactly equal to $\mathfrak{E}(G)^{W(G)}$. Clearly, characters which factor as elements of $\mathfrak{R}(G)$ also factor in $\mathfrak{E}(G)$, but there are numerous examples showing the converse implication does not hold. In fact, we exploit the existence of extra factors in $\mathfrak{E}(G)$ to deduce the existence of the series of Γ-factorizations, ^{LS}-factorizations, and Γ^{LS}-factorizations in $\mathfrak{R}(G)$.

As we will see, each class of factorization has a natural relationship to the root lattice of G, and thus affords us a geometrical connection to the factorization problem. Indeed, several of our proofs rely on what is known about the geometry of the root lattices. In addition, in the case of ordinary and exotic factorizations of characters of G, the geometrical pictures are further related to maps on the corresponding classifying spaces BG. Indeed it seems that these factorizations should be related to certain tensor products of vector bundles over BG, although we have not yet investigated the exact relationship.

Finally, it is our belief that the factorizations obtained in this dissertation are complete among irreducible representations. There do not appear to be any further considerations in $\mathfrak{E}(G)$ which might lead to series of factorizations; this indicates with strong probability that the various infinite series of factorizations we discuss form a complete list. However, we have thusfar been unsuccessful in attempts to show that no sporadic factorizations appear among the irreducible characters.

Our approach is as follows: after discussing the background material and introducing the relevant concepts, in §2.1.1 through §2.2.3 we begin the main body by looking at examples of factorizations in some low-rank groups. Once we have begun to see general patterns, in §3.1.1 we will prove the existence of the ordinary and exotic factorizations. In the first part of §3.2 we consider some factorization patterns from our observations which are not covered by the Theorems in §3.1.1. Finally we combine the factorizations from §3.1.1 and the first part of §3.2 with the fact that we are working in a UFD to deduce additional factorizations which would not be apparent just from our observations of the tables of factorizations which we construct. Finally, in §4.2 we explore the appearances of these factorization results among recursively defined sequences of integers with a divisibility property and discuss further avenues of study.

1.1 Background Material

1.1.1 Basics of Lie Groups

Throughout the course of this paper a Lie group G is understood to be compact, connected, simple, semisimple, and simply-connected and all representations of G are complex representations. When G is arbitrary, the variable n will be used exclusively in reference to the rank.

When dealing with a specific group we use the standard Dynkin names $\mathcal{A}_n$, $\mathcal{B}_n$, etc. The groups $\mathcal{B}_n$, $\mathcal{C}_n$, $\mathcal{F}_4$, and $\mathcal{G}_2$ are $nonsimply-laced$, refering to the fact that their associated Dynkin diagrams contain double or triple edges.

Associated to G are several related objects, its maximal torus $\mathbb{T}(G) = \mathbb{T}$, Root System $\mathbf{R_G} \subset \widetilde{\mathbb{T}}$, and Weyl Group $W(G)$, being the most important for our purposes. Although $\mathbb{T}$ is not unique, every element of G lies in some maximal torus and any two maximal tori of G are G-conjugate to one another, so for our purposes it will be sufficient to consider $\mathbb{T}$ as fixed. Although it is important in general, we do not use many facts about the Lie algebra $\mathfrak{g}$ associated to G, other than the fact that it forms a vector space of dimension $\dim(G)$. As a maximal torus is itself a Lie group, it possesses a Lie algebra as well, which is equal to its universal cover and which we denote $\widetilde{\mathbb{T}}$. Although we shall not need the exponential map defined on $\mathfrak{g}$, we will often use its dual map which we denote exp:

$$\exp : \widetilde{\mathbb{T}}^{\vee} \to Hom(\mathbb{T}, U(1))$$

Given $\mathbf{R_G}$ one can always choose n linearly independent roots $\{\alpha_1, \ldots, \alpha_n\}$, the *simple* roots, such that all other roots are either nonnegative or nonpositive integer combinations of the simple roots. This decomposes $\mathbf{R_G}$ into $\mathbf{R_G^+} \sqcup \mathbf{R_G^-}$, the sets of *positive* and *negative* roots respectively. The lattice formed by all integer combinations of the simple roots is the root lattice of G; the root system is a certain subset of this lattice. The root lattice sits inside $\widetilde{\mathbb{T}}$ as the kernel of the exponential map $\widetilde{\mathbb{T}} \to \mathbb{T}$.

Definition 1.1.1. *The* **characteristic** *of a simple Lie Group G, is the square of the ratio of the length of the longest root to the length of the shortest root in $\mathbf{R_G}$.*

Thus $\mathcal{A}_n$, $\mathcal{D}_n$, and $\mathcal{E}_{6,7,8}$ have characteristic 1, $\mathcal{B}_n$, $\mathcal{C}_n$, and $\mathcal{F}_4$ have characteristic 2, and $\mathcal{G}_2$ has characteristic 3. The variable q will be used to refer to the characteristic.

Dual to the root lattice is the weight lattice, and the dual basis to the simple roots are the fundamental weights $\omega_j := \alpha_j^\vee$. In particular, the weights correspond to linear functionals on the Lie algebra of a given maximal torus in G. We use a coordinate system on the weight lattice in which ω_j is simply written as the j^{th} coordinate vector so that a weight such as $2\omega_1 + \omega_3$ would be written as $[2, 0, 1, 0, \ldots]$ for example. There is a natural partial order on the weight lattice induced by the simple roots, see [BD] or [FH] for details.

An arbitrary weight will usually be denoted $[I]$ which is shorthand notation for:

$$[I] = \sum_{k=1}^{n} I_k \omega_k$$

When working with indeterminate weights $[I]$ it is assumed that all $I_k \geq 0$, i.e. $[I]$ is a *dominant weight*. Most of the time the group to which a weight is associated is understood from context; however in a few cases we will be working with multiple groups, so in those cases to reduce confusion we may write $[I]_G$ to specify the associated group.

One particular weight which appears often in various contexts is the weight $\sum_{k=1}^{n} \omega_k$ which is the weight dual to the half-sum of all roots in $\mathbf{R}_{\mathbf{G}}^{+}$. We denote this weight $[\rho]$ for convenience; notation such as $[I+\rho]$ is then understood to be the weight obtained by adding 1 to each I_k.

For many of our calculations it is more convenient to use $\rho - shifted$ indexing which we denote with a hat above the indexed object S and which is defined by:

$$\widehat{S}_{[I+\rho]}(G) := S_{[I]}(G)$$

Using the map exp introduced earlier, we define:

$$X_j := \exp(\omega_j)$$

The X_j are thus particular elements of $Hom(\mathbb{T}, U(1))$ and are most conveniently viewed as defining a set of coordinates on $\mathbb{T}$; given $g \in G$, the X_j indicate where g lies in $\mathbb{T}$.

For a given G, the set of isomorphism classes of finite dimensional representations of G forms a semiring under the natural operations $\oplus$ and $\otimes$; one may complete this semiring to a ring by formally introducing the operation of $\ominus$ defined such that if $C = A \ominus B$ then $A = B \oplus C$. The resulting ring, denoted $\mathfrak{R}(G)$ is the Virtual Representation Ring of G; elements of $\mathfrak{R}(G)$ in general are not isomorphic to any G-module. An element of $\mathfrak{R}(G)$ which corresponds to a G-module will be referred to as *honest* while an element which either does not correspond to any G-module or whose status is unknown will be referred to as *virtual*.

Note that an honest representation of G need not involve only $+$ and $\cdot$ operations as an expression in $\mathfrak{R}(G)$. For example, denoting the k-dimensional irreducible representation of $SU(2)$ by $\pi_{[k-1]}$, as elements of $\mathfrak{R}(A_1)$ one has:

$$\pi_{[2]} = \pi_{[1]} \cdot \pi_{[1]} - \pi_{[0]}$$

We use the variable π when considering a representation as an element of $\mathfrak{R}(G)$. Each representation $\pi(G)$ has an associated set of weights, and with respect to the partial order on weights, each irreducible representation (or irrep for short) of G possesses a unique *highest weight*, i.e. a dominant weight $[I]$ which is greater than all other weights of the representation; this highest weight appears with multiplicity 1. We index irreps by their highest weights, such as $\pi_{[I]}(G)$ which is an irrep of G with highest weight $i_1\omega_1 + i_2\omega_2 + \ldots$. We will occasionally omit the reference to the group after the representation name when it is clear from context which group we are dealing with.

When the highest weight is a fundamental weight ω_j, we will usually denote the irreducible representation simply by $\pi_{\omega_j}(G)$; such representations are often referred to as *fundamental representations* of G. It is well-known that $\mathfrak{R}(G)$ is isomorphic to a polynomial ring with integer coefficients and generated by the fundamental representations. Thus, when dealing with elements of $\mathfrak{R}(G)$ we usually replace the bulky notation $\pi_{\omega_j}(G)$ by a formal variable V_j and work with elements of $\mathfrak{R}(G)$ as elements of the isomorphic ring $\mathbb{Z}[V_1, \ldots, V_n]$.

The second important ring for our purposes is the Virtual Character Ring of $\mathbb{T}(G)$, which we denote $\mathfrak{E}(G)$. $\mathfrak{E}(G)$ is isomorphic to a Laurent polynomial ring generated by the X_j:

$$\mathfrak{E}(G) = \mathbb{Z}[X^*(\mathbb{T})] \cong \mathbb{Z}[X_j, X_j^{-1}]_{j=1}^n$$

We associate each weight of a given representation π to a monomial formed from the X_j by applying the map exp to the weight; in this way we naturally associate each representation π to a corresponding element $\chi \in \mathfrak{E}(G)$. If $\pi = \pi_{[I]}(G)$ is an irrep, then this corresponding element is denoted $\chi_{[I]}(G)$. Recalling that each $g \in G$ lies in some $\mathbb{T}$, evaluation of $\chi(g)$ returns the value of the character of π at g; furthermore, evaluation of the monomial terms of χ returns the individual eigenvalues of g in the representation π (we thus sometimes refer to $\mathfrak{E}(G)$ as the Eigenvalue Ring of G). In general, we will use χ to refer to the character of a representation when we are considering its factorization in $\mathfrak{E}(G)$. If π is further named by a subscript then the corresponding element χ also carries the same subscript.

The *dimension* of an arbitrary representation $\pi \in \mathfrak{R}(G)$ is obtained by setting $V_k = \dim(\pi_{\omega_k}(G))$ for all k; when the polynomial corresponds to an honest representation this notion of dimension corresponds to the usual dimension of the representation. Similarly, for an element $\chi \in \mathfrak{E}(G)$, the dimension is obtained by specializing $X_k = 1$ for all $1 \leq k \leq n$.

The Weyl Group of G, denoted $W(G)$, is the Coxeter group generated by reflections in the hyperplanes orthogonal to the roots of G, thus there is a natural $W(G)$-action on the universal cover of $\mathbb{T}$. This action induces corresponding actions on the weight lattice, $\mathbb{T}$, and on $\mathfrak{E}(G)$.

As it is a Coxeter group, $W(G)$ has a nontrivial homomorphism $sgn : W(G) \to \mathbb{Z}/2\mathbb{Z} \subset U(1)$ which sends all the Coxeter generators to -1. This is the sign representation and we will use the notation $(-1)^w$ to denote its action:

$$(-1)^w := sgn(w)$$

For $g \in \mathfrak{E}(G)$ and for $w \in W(G)$, w-action on g is denoted $w(g)$ and is given by:

$$w(g) = g(w(X_1), \ldots, w(X_n))$$

Definition 1.1.2. *Let $S = \{s_i\}_{i \in \mathcal{I}}$ be a set of elements of $\mathfrak{E}(G)$.*

S is $\mathbf{W(G)-}$**alternating** *if $S = \{(-1)^w \cdot w(s_i)\}_{i \in \mathcal{I}}$ for each $w \in W(G)$. An element $s \in \mathfrak{E}(G)$ is a $W(G)$-alternating element if $s = (-1)^w \cdot w(s)$ for all $w \in W(G)$.*

S is $\mathbf{W(G)-}$**symmetric** *if $S = \{w(s_i)\}_{i \in \mathcal{I}}$ for each $w \in W(G)$. An element $s \in \mathfrak{E}(G)$ is a $W(G)$-symmetric element if $s = w(s)$ for all $w \in W(G)$.*

Since monomials in $\mathfrak{E}(G)$ naturally correspond to weights of G counted with multiplicity, by abuse of notation we will also sometimes refer to a set of weights as $W(G)$-alternating or symmetric.

$W(G)$-alternating sets and $W(G)$-alternating elements are in 1-1 correspondence. The correspondence is given by multiplying together the elements of an alternating set to obtain an alternating element; conversely, factoring an alternating element gives an alternating set. The analogous statements hold for $W(G)$-symmetric sets and elements.

The product of the elements of a $W(G)$-alternating set is a $W(G)$-alternating element of $\mathfrak{E}(G)$ and likewise the product of the elements of a $W(G)$-symmetric set is a $W(G)$-symmetric element of $\mathfrak{E}(G)$. The constant polynomials are the only elements of $\mathfrak{E}(G)$ which are simultaneously $W(G)$-alternating and $W(G)$-symmetric.

$W(G)$ permutes the weights of any representation, so characters are $W(G)$-symmetric elements of $\mathfrak{E}(G)$ and in fact one has $\mathfrak{E}(G)^{W(G)} \cong \mathfrak{R}(G)$. In particular, every $W(G)$-symmetric element of $\mathfrak{E}(G)$ is the character of some element of $\mathfrak{R}(G)$, a fact which we will often use.

As every weight of G lies in the $W(G)$-orbit of a unique dominant weight, for any non-dominant weight $[J]$, one can 'factor' $[J]$ as $w \circ [I]$ for some $w \in W(G)$ and $[I]$ a dominant weight of G. The dominant weight $[I]$ is unique in this factorization, but the Weyl group element w need not be unique. Using this factorization, one extends the concept of highest weight irreps to all weights as follows:

$$\pi_{[J]}(G) = (-1)^w \pi_{[I]}(G)$$

1.1.2 Classifying Spaces

As mentioned in the introduction, associated to each G is a classifying space BG. We will not use BG directly in our results, but some results of Adams' et. al. on BG will be relevant to our later discussions so we introduce them now. In [AD1], Adams and Mahmud consider the problem of determining maps between classifying spaces of compact Lie Groups. Their result uses the notion of an admissible map which they define as:

Definition 1.1.3. *Let G and G' be arbitrary compact semisimple Lie Groups with fixed maximal torii $\mathbb{T}(G)$ and $\mathbb{T}(G')$ respectively. Let $\widetilde{\mathbb{T}}(G)$ be the universal cover of $\mathbb{T}(G)$ and similarly for G'. A linear map $\tau : \widetilde{\mathbb{T}}(G) \to \widetilde{\mathbb{T}}(G')$ is* **admissible** *if for every $w \in W(G)$ there is $w' \in W(G')$ such that:*

$$\tau \circ w = w' \circ \tau$$

The definition used in [AD1] assumes slightly more, but the extra assumptions are not necessary for our purposes. They then proceed to show:

Theorem 1.1.1. *(Adams, Mahmud) There is a 1-1 correspondence between admissible maps $\tau : \widetilde{\mathbb{T}}(G) \to \widetilde{\mathbb{T}}(G')$ and maps $f : BG \to BG'[\frac{1}{n}]$ (with certain restrictions* *on n which depend on τ).*

The conditions on n do not affect our results so we will ignore them.

The cases which turn out to be relevant to our work are the cases when $G' = G$ or $G' = G^*$ where G^* denotes the Cartan dual of G; i.e. the simply connected Lie group which has the same Dynkin diagram as G but with all arrows reversed (in particular $G \ncong G^*$ iff $G = \mathcal{B}_n$ or $\mathcal{C}_n$ with $n \geq 3$). In order to understand their results in these cases as well as several of our later results, we first introduce the Adams' operations ψ^m_G (or simply ψ^m when G is clear from context) which are defined as follows. Letting χ be the character of $\pi \in \mathfrak{R}(G)$ and let $g \in G$, ψ^m is an endomorphism of $\mathfrak{E}(G)^{W(G)}$ whose action on characters is given by:

$$\psi^m \chi(g) = \chi(g^m)$$

This extends to an endomorphism of all of $\mathfrak{E}(G)$ by:

$$\psi^m(X_j) = X_j^m$$

The corresponding action on the weight lattice multiplies every coordinate of a given weight $[I]$ by m; the notation we will use for this action will simply be $m[I]$. Extending this action we have that ψ^m_G acts on all of $\widetilde{\mathbb{T}}(G)$ by dilation by a factor of m.

As already noted, applying ψ^m clearly does not affect $W(G)$-invariance of a character (since ψ^m corresponds to an admissible map), so ψ^m also induces an endomorphism $\Psi^m : \mathfrak{R}(G) \to \mathfrak{R}(G)$ defined by the condition that if χ is the character of $\pi \in \mathfrak{R}(G)$, then $\Psi^m \pi$ is the element of $\mathfrak{R}(G)$ whose character is $\psi^m \chi$.

Using the Adams' operations, the admissible maps $\tau : G \to G$ and $\tau : G \to G^*$ are classified as follows:

Theorem 1.1.2. *(Adams, Mahmud) The ordinary admissible maps $\widetilde{\mathbb{T}}(G) \to \widetilde{\mathbb{T}}(G)$ are dilation of $\widetilde{\mathbb{T}}(G)$ by a factor of m with $m \in \mathbb{Q}$; when $m \in \mathbb{N}$ these correspond to the actions of Adams' operations on $\widetilde{\mathbb{T}}(G)$. In addition, when G is not simply-laced there is a unique (up to $W(G)$-equivalence) exotic admissible map $\varepsilon_G : \widetilde{\mathbb{T}}(G) \to \widetilde{\mathbb{T}}(G^*)$ such that:*

$$\begin{aligned} \varepsilon_{\mathcal{C}_n} \circ \varepsilon_{\mathcal{B}_n} &= \psi^2_{\mathcal{B}_n} \\ \varepsilon_{\mathcal{B}_n} \circ \varepsilon_{\mathcal{C}_n} &= \psi^2_{\mathcal{C}_n} \\ \varepsilon_{\mathcal{F}_4} \circ \varepsilon_{\mathcal{F}_4} &= \psi^2_{\mathcal{F}_4} \\ \varepsilon_{\mathcal{G}_2} \circ \varepsilon_{\mathcal{G}_2} &= \psi^3_{\mathcal{G}_2} \end{aligned} \tag{1.1.1}$$

Remark. Note that for $G = \mathcal{B}_2$, $\mathcal{F}_4$, and $\mathcal{G}_2$, $G \cong G^*$, so in particular $\varepsilon_G = \varepsilon_{G^*}$ for these cases; but the exotic maps implied by the theorem in these cases are non-trivial by Condition 1.1.1. In describing an exotic admissible map, it is enough to describe its action on the simple roots since they span $\widetilde{\mathbb{T}}$.

Although Theorem 1.1.2 allows m to be any rational number in an ordinary admissible maps, on the level of characters the ordinary admissible maps correspond to the ψ^m_G when $m \in \mathbb{Z}$ so these are the only ordinary admissible maps we will be interested in.

The remaining maps, which Adams calls ε for 'exotic', do not have as simple description as the ordinary Adams' maps; for now we only note that they induce corresponding ring homomorphisms $\mathfrak{E}(G) \to \mathfrak{E}(G^*)$ which, in keeping with Adams' description of these as exotic maps, we denote ξ_G (or simply ξ when G is clear from context).

Remark. If q is the characteristic of G, the maps ξ satisfy an analogue of (1.1.1):

$$\xi_{G^*} \circ \xi_G = \psi_G^q \tag{1.1.2}$$

As before, since ξ comes from an admissible map, there is a corresponding ring homomorphism $\Xi : \mathfrak{R}(G) \to \mathfrak{R}(G)$ defined analogously to Ψ^m. There is also a corresponding map $\epsilon = \epsilon_G$ on weights whose action we denote $\epsilon[I]$. The descriptions of the actions of ξ, Ξ and $\epsilon[I]$ will be dealt with as they arise in our work.

Finally, we note that besides the cases of maps $BG \to BG$ and $BG \to BG^*$, [AD1] and [AD2] study numerous other examples of maps $BG \to BG'$. Based on the fact that our factorization results are connected to the cases above, these other maps may offer a promising starting point for finding related factorization results.

1.1.3 Useful Theorems and Algorithms for Calculations

In order to effectively work with the irreps of G, we need efficient methods of calculating their images in $\mathfrak{R}(G)$ and $\mathfrak{E}(G)$. To do this, we make use of Klimyk's

Formula in $\mathfrak{R}(G)$ and the Weyl Character Formula in $\mathfrak{E}(G)$.

Theorem 1.1.3. *(Klimyk) Let $\pi_{[J]}(G)$ be an irrep of dimension d and let $\{\alpha_p\}_{p=1}^{d}$ be the set of weights of $\pi_{[J]}(G)$. Then for an arbitrary irrep $\pi_{[I]}(G)$ one has:*

$$\pi_{[I]}(G) \otimes \pi_{[J]}(G) \cong \bigoplus_{p=1}^{d} \pi_{[I+\alpha_p]}(G)$$

Proof. See [Kl] for details. $\square$

Remark. For some p, $[I + \alpha_p]$ may not be a dominant weight which means that the above direct sum may actually be of the form $\Pi_1(G) \ominus \Pi_2(G)$ with each $\Pi_i(G)$ honest when expanded out; however in the case that both $[I]$ and $[J]$ are dominant weights, it is known that $\Pi_2(G)$ is always a subrepresentation of $\Pi_1(G)$ so that $\pi_{[I]}(G) \otimes \pi_{[J]}(G)$ is nonetheless honest.

In theory, Theorem 1.1.3 allows one to recursively compute the polynomials of irreps starting from the polynomials of the fundamental irreps (which are monomials, hence easy to deal with). This recursive method works well for groups of low rank and was the method employed in calculating the polynomials for irreps of $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{C}_3$ and $\mathcal{G}_2$ used in our work. For larger ranks the formula quickly becomes too cumbersome to be practical because the fundamental representations have too many weights and we instead use a trick involving the Weyl Character Formula.

Theorem 1.1.4. *(Weyl Character Formula) For* $[I]$ *a weight of* G*, let* $E_{[I]}(G) \in \mathfrak{E}(G)$ *be given by:*

$$E_{[I]}(G) := \sum_{w \in W(G)} (-1)^w \exp(w \circ [I])$$

The irreducible character $\chi_{[I]}(G)$ *is given by:*

$$\chi_{[I]}(G) = \frac{E_{[I+\rho]}(G)}{E_{[\rho]}(G)}$$

Proof. See [BD] for details. $\square$

Remark. From its definition, it is clear that $E_{[I]}(G)$ is $W(G)$-alternating for any weight $[I]$.

Besides explicitly computing characters, the Character Formula allows us to easily compute both the image of $\pi_{[I]}(G)$ in $\mathfrak{R}(G)$ as well as the decomposition into irreducible summands of a given representation as outlined in the next two Algorithms. In both, if $\{Y_i\}$ is a sequence of values, we use the notation $Y_\infty = Y_k$ for $k >> 0$. All such sequences we will use will be seen to stabilize in finite time so that Y_∞ makes sense.

Algorithm 1.1.1. *(Computation of the element* $\pi(G) \in \mathfrak{R}(G)$ *corresponding to a given character* $\chi(G) \in \mathfrak{E}(G)$*) Compute the fundamental characters of* G *using the Character Formula. Set* $X_0 := \chi(G)$*. Given* X_i*, choose a highest weight* $[J^i]$ *of* X_i *and let its multiplicity be* μ_i*; for* X_0*, clearly* $[J^0] = [I]$ *and* $\mu_0 = 1$*. In general the* X_i *may not have a unique highest weight, but as they are Laurent polynomials they* *possess only finitely many terms, so they must possess at least one dominant weight which is maximal in the partial order. Recursively define:*

$$X_{i+1} := X_i - \mu_i \prod_{k=1}^{n} [\chi_{\omega_k}(G)]^{J_k^i}$$

Since $\pi_{[I]}(G)$ *is an element of* $\mathfrak{R}(G)$*, this process eventually stabilizes; thus* $X_\infty = \mu_\infty = 0$ *and the image of* $\pi_{[I]}(G)$ *in* $\mathfrak{R}(G)$ *is given by:*

$$\sum_{i=0}^{\infty} \left(\mu_i \cdot \prod_{k=1}^{n} V_k^{J_k^i} \right)$$

Remark. Note that Algorithm 1.1.1 uses unshifted characters, so care must be taken to make sure inputs are unshifted before applying it.

Remark. The usefulness of Algorithm 1.1.1 will be seen in §2.2.2 when we need to check whether certain polynomials factor to verify predictions.

Algorithm 1.1.2. *(Decomposition of Arbitrary Characters into Irreducibles) If* $\Pi(G)$ *is a representation such its image in* $\mathfrak{R}(G)$ *or its character in* $\mathfrak{E}(G)$ *are known, then the decomposition of* $\Pi(G)$ *into irreducible summands is computable without explicitly calculating the irrep polynomials.*

If Π *is known as an element of* $\mathfrak{R}(G)$*, use the fundamental characters to compute its character in* $\mathfrak{E}(G)$*. Once the character of* Π *is known, multiply this character by* $E_{[\rho]}(G)$ *to obtain an element* $E(\Pi) \in \mathfrak{E}(G)$*. Set* $E_0 := E(\Pi)$ *and let* $[J^0]$ *be a highest weight of* $E(\Pi)$ *with multiplicity* ν_0*. Recursively, as in the proof of Algorithm 1.1.1, choose a highest weight* $[J^i]$ *of* E_i *with multiplicity* μ_i *and define:*

$$E_{i+1} := E_i - \mu_i \cdot E_{[J^i]}(G)$$

As Π *decomposes into a finite number of irreducibles,* $E_\infty = \mu_\infty = 0$ *and the desired decomposition of* Π *into irreducibles is:*

$$\Pi = \bigoplus_{i=0}^{\infty} \mu_i \widehat{\pi}_{[J^i]}(G)$$

Remark. Note that the irreducible summands appearing in the final result are $\widehat{\pi}_{[J^i]}(G)$, not $\pi_{[J^i]}(G)$ as one might at first expect. This shifting occurs precisely because the denominator in the Character Formula is $E_{[\rho]}(G)$ instead of $E_{[0]}(G)$.

Remark. While the characters of the irreducible summands may grow arbitrarily large, each $E_{[J^i]}$ only contains $|W(G)|$ terms, hence the growth of $|E_i|$ is at most linear (and furthermore eventually decays to 0). This slow growth and bounded overall size makes it feasible to employ a computer to carry out decompositions of large characters without quickly running out of memory.

Remark. In applying Algorithm 1.1.2 it is often possible to work with a variable m in the expressions for the $E_{[I]}(G)$, thereby allowing one to prove results for arbitrary m instead of performing case-by-case analyses. In such cases, the highest weight $[J^i]$ is obtained by ignoring the m-dependence of the weights which appear in E_i, choosing the highest weight, then adding the m-dependence back on to this weight.

Remark. The usefulness of Algorithm 1.1.2 will be seen in §3.2 when certain characters are easy to compute, and the decompositions of the corresponding virtual representations are desired. For example, it affords us the ability to use symbolic manipulation to simultaneously decompose the infinite classes of ^{LS}-factors in §3.2.2, whose characters are easily computed, into irreducible components. Such decompositions would be tedious if done using characters as they could only be done one factor at a time and one would have no guarantee that the patterns obtained hold forever. It also allows one to compute the decompositions into irreducibles of the series of Γ-factors and Γ^{LS}-factors introduced in §3.2.1 and §3.2.3 respectively, although in these cases the decompositions must be computed individually instead of simultaneously.

Related to Theorem 1.1.4 and central to much of our discussion in §3.2 will be the Weyl Denominator Formula which gives a description of $E_{[\rho]}(G)$, as well as some of its corollaries. We state the Denominator Formula here, and defer its corollaries until the sections in which they are needed.

Definition 1.1.4. *Let u_j denote the vector formed by the j^{th} row of the Cartan matrix of G. For $r = \sum_{i=1}^{n} k_i(r)\alpha_i$ an arbitrary element of the root lattice of G, define v to be the vector-valued function on the root lattice of G given by:*

$$v(r) := \frac{1}{2}\sum_{i=1}^{n} k_i(r)u_i$$

Using this, we further define X^v to be the $\mathfrak{E}(G)$-valued function on the root lattice of G given by:

$$X^{v(r)} := \prod_{i=1}^{n} X_i^{v(r)_i}$$

Remark. The function v is chosen so that the weight $\omega_r := \sum_{i=1}^{n} v(r)_i \omega_i$ is the dual weight to $\frac{1}{2}r$. In particular, if r is the sum of all positive roots of G, then $\omega_r = \rho$.

Theorem 1.1.5. *(Weyl Denominator Formula)*

In $\mathfrak{E}(G)$, $E_{[\rho]}(G)$ factors as:

$$E_{[\rho]}(G) = \prod_{r \in \mathbf{R}_{\mathbf{G}}^{+}} (X^{v(r)} - X^{-v(r)})$$

Proof. See [FH], Lemma 24.3 for details. □

Remark. It is clear from this factorization that the factors of $E_{[\rho]}(G)$ form an $W(G)$-alternating set.

1.2 Fundamental Weight Labelling Scheme

The Dynkin Diagram of G gives information about how the simple roots of G lie relative to one another in the root system. In particular, each node of the diagram corresponds to a simple root and hence to a fundamental weight as well. The following are the labellings of the fundamental weights which we shall use when dealing with the nonsimply-laced groups. A filled in node indicates the fundamental weight corresponds to a short simple root, while a hollow node indicates the weight corresponds to a long simple root.

1.3 Summary of Commonly Used Notations

G	A an arbitrary connected, simply-connected, semisimple simple Lie Group of rank n
$W(G)$	The Weyl Group of G
$\mathfrak{R}(G)$	The Virtual Representation Ring of G
$\mathfrak{E}(G)$	The Eigenvalue Ring of G

ω_k	The k^{th} fundamental weight of a Lie Group; the index corresponds to a given indexing of the nodes of the Dynkin Diagram
X_k	The generator $\exp(\omega_k)$ of $\mathfrak{E}(G)$
$[I]$ or $[I_1, \ldots, I_n]$	The weight $\sum_{k=1}^{n} I_k \omega_k$
$[\rho]$	The weight $[1, \ldots, 1]$ dual to the half-sum of positive roots
$m[I]$	The weight obtained from $[I]$ by the induced action of ψ^m
$\epsilon[I]$	The weight of G^* obtained from $[I]$ by the induced action of ξ
$w \circ [I]$	The weight obtained by the action of $w \in W(G)$ on $[I]$
$E_{[I]}(G)$ or $E_{[I]}$	The function $\sum_{w \in W(G)} (-1)^w \exp(w \circ [I])$
$\pi_{[I]}(G)$ or $\pi_{[I]}$	The irrep of G with highest weight $[I]$
$\chi_{[I]}(G)$ or $\chi_{[I]}$	The character of $\pi_{[I]}$, given by $\chi_{[I]} = \frac{E_{[I+\rho]}}{E_{[\rho]}}$
$\pi_{\omega_k}(G)$ or π_{ω_k}	The fundamental irrep of G with highest weight ω_k
V_k	Notation for π_{ω_k} as a generator of $\mathfrak{R}(G)$
$\chi_{\omega_k}(G)$ or χ_{ω_k}	The character of π_{ω_k}, sometimes referred to as a fundamental character

Chapter 2

Examples of Factorization of Irreps

2.1 Simply-Laced Groups

We begin our work by looking at factorizations of irreps of $\mathcal{A}_1$ and $\mathcal{A}_2$ where the Weyl Groups and irreps are small enough to be able to easily calculate a number of results using the Weyl Character Formula and Klimyk's Formula. The ultimate result is the appearance of what we will call ordinary factorizations (whose existence in all Lie Groups we prove in §3.1.1) and Γ factorizations (whose existence in all Lie Groups we prove in §3.2). Our work shows that working just in $\mathcal{A}_1$ does not accurately exhibit the nature of ordinary factorizations while working in $\mathcal{A}_2$ does not clearly exhibit the nature of Γ-factorizations.

2.1.1 The Lie Group $\mathcal{A}_1$

We start by examining the irreducible representations of $\mathcal{A}_1 = SU(2)$. The weights of $\mathcal{A}_1$ form a 1-dimensional lattice generated by the fundamental weight $\omega_1 := \omega_1(\mathcal{A}_1)$ and parametrized by $\mathbb{Z}$ so that irreducible representations are parametrized by a single nonnegative integer k so that the irreps are of the form $\pi_{[k]}(\mathcal{A}_1)$ with highest weight $k\omega_1$ and the corresponding character is $\chi_{[k]}(\mathcal{A}_1)$. As there is only one Lie Group of rank 1, there is no chance of ambiguity as to the Lie Group, so we drop the reference to $\mathcal{A}_1$ throughout this section.

By the Weyl Character formula, the dimension of $\pi_{[k]}$ is $(k+1)$. The fundamental representation $\pi_{[1]}$ of dimension 2 may be explicitly realized as the standard action of the group of unit quaternions on $\mathbb{C}^2$; for arbitrary $g = a + b\hat{\imath} + c\hat{\jmath} + d\hat{k}$ this representation is given by:

$$\pi_{[1]}(g) = \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$

The character of this representation is therefore:

$$\chi_{[1]}(g) = 2 \cdot \mathfrak{Re}(g) = 2a$$

Note that a is itself a function of g, so that the polynomial $2a$ completely describes the character $\chi_{[1]}(g)$ for any $g \in \mathcal{A}_1$. We therefore drop the reference to g when describing characters in general.

Klimyk's Formula (Theorem 1.1.3) gives the following relationship between irreducible representations of $\mathcal{A}_1$:

$$\pi_{[n]} \otimes \pi_{[1]} \cong \pi_{[n+1]} \oplus \pi_{[n-1]}$$

Replacing each representation by its character and rearranging we thus obtain a recurrence relation for the characters:

$$\chi_{[n+1]} = \chi_{[n]} \cdot \chi_{[1]} - \chi_{[n-1]}$$

Since $\chi_{[1]}(g) = 2a$, by induction $\chi_{[n]}$ is of degree n in $\mathbb{Z}[2a]$. The first few of these are given below:

n	$\chi_{[n]}$	n	$\chi_{[n]}$
0	1	5	$32a^5 - 32a^3 + 6a$
1	$2a$	6	$64a^6 - 80a^4 + 24a^2 - 1$
2	$4a^2 - 1$	7	$128a^7 - 192a^5 + 80a^3 - 8a$
3	$8a^3 - 4a$	8	$256a^8 - 448a^6 + 240a^4 - 40a^2 + 1$
4	$16a^4 - 12a^2 + 1$	9	$512a^9 - 1024a^7 + 672a^5 - 160a^3 + 10a$

Table 2.1: Irreducible Characters $\chi_{[n]}$ for small n

The polynomials in Table 2.1 are easily recognized to be Chebyshev polynomials of the second kind, so from general theory of Chebyshev polynomials we already know each $\chi_{[n]}$ factors in $\mathbb{Z}[a]$. However we will ignore this fact and instead use a more general approach which will apply to other compact semisimple Lie groups. First we try factoring the first few $\chi_{[n]}$ in $\mathbb{Z}[\chi_{[1]}]$:

n	Factorization of $\chi_{[n]}$
0	1
1	$\chi_{[1]}$
2	$(\chi_{[1]} - 1) \cdot (\chi_{[1]} + 1)$
3	$\chi_{[1]} \cdot (\chi_{[1]}^2 - 2)$
4	$(\chi_{[1]}^2 - \chi_{[1]} - 1) \cdot (\chi_{[1]}^2 + \chi_{[1]} - 1)$
5	$\chi_{[1]} \cdot (\chi_{[1]} - 1) \cdot (\chi_{[1]} + 1) \cdot (\chi_{[1]}^2 - 3)$
6	$(\chi_{[1]}^3 + \chi_{[1]}^2 - 2\chi_{[1]} - 1) \cdot (\chi_{[1]}^3 - \chi_{[1]}^2 - 2\chi_{[1]} + 1)$
7	$\chi_{[1]} \cdot (\chi_{[1]}^2 - 2) \cdot (\chi_{[1]}^4 - 4\chi_{[1]}^2 + 2)$

Table 2.2: Factors of $\chi_{[n]}$ in $\mathbb{Z}[\chi_{[1]}]$

From this table we already see a pair of interesting features. Firstly, all $\chi_{[n]}$ factors nontrivially for small n, and indeed we will see that $\chi_{[n]}$ factors in general. Secondly, some factors such as $\chi_{[1]}$,$\chi_{[1]} - 1$ and $\chi_{[1]} + 1$ appear multiple times in the table. This phenomenon will also be explained shortly.

To understand why the $\chi_{[n]}$ always factor, we approach the problem from another angle. Recall that a character is really a sum of eigenvalues. So let us examine the eigenvalues of $g = a + b\hat{\imath} + c\hat{\jmath} + d\hat{k}$ in $\pi_{[1]}$; a short computation shows that they are $a \pm \sqrt{a^2 - 1}$ and the determinant condition on $\mathcal{A}_1$ shows that they are inverses to one another. Setting $X_1 := a + \sqrt{a^2 - 1}$ we can thus also express the fundamental character as:

$$\chi_{[1]}(g) = X_1 + X_1^{-1}$$

Although the polynomial 2a was derived from the explicit action of $SU(2)$ on $\mathbb{C}^2$, with respect to Lie theory in general it is not a natural construction; in particular, it does not reflect $W(\mathcal{A}_1)$-invariance in the character. However, the alternative character $X_1 + X_1^{-1}$ does reflect this invariance since the action of the nontrivial element $\sigma \in W(\mathcal{A}_1)$ on the weights of $\mathcal{A}_1$ is given by $\sigma \circ [k] = [-k]$. Indeed, $X_1 + X_1^{-1}$ is the character predicted by the Weyl Character Formula applied to $\mathcal{A}_1$:

$$\begin{aligned}\chi_{[n]} &= \frac{E_{[n+1]}(\mathcal{A}_1)}{E_{[1]}(\mathcal{A}_1)} \\ &= \frac{X_1^{n+1} - X_1^{-n-1}}{X_1 - X_1^{-1}}\end{aligned}$$

From this point of view, the factorizations observed earlier are easily understood; to explain them we first make a definition.

Definition 2.1.1. *Let ζ_d denote a primitive d^{th} root of unity. The d^{th} homogenous cyclotomic polynomial in X and Y is defined as:*

$$\begin{aligned}\Phi_d(X, Y) &:= Prim(X^d - Y^d) \\ &= \prod_{\substack{1 \le k \le d \\ gcd(k,d)=1}} (X - \zeta_d^k Y)\end{aligned}$$

Now notice that $E_{[n+1]}$ is of the form $X_1^{n+1} - (X_1^{-1})^{n+1}$; it therefore factors into homogenized cyclotomic factors in X and X^{-1}:

$$E_{[n+1]} = \prod_{d|(n+1)} \Phi_d(X_1, X_1^{-1})$$

Thus we have the following factorizations of the $\chi_{[n]}$:

n	$\chi_{[n]}$
0	-
1	$\Phi_2(X_1, X_1^{-1})$
2	$\Phi_3(X_1, X_1^{-1})$
3	$\Phi_2(X_1, X_1^{-1})\Phi_4(X_1, X_1^{-1})$
4	$\Phi_5(X_1, X_1^{-1})$
5	$\Phi_2(X_1, X_1^{-1})\Phi_3(X_1, X_1^{-1})\Phi_6(X_1, X_1^{-1})$
$\ldots$	$\ldots$

Table 2.3: Cyclotomic Factorization of $\chi_{[n]}$ in $\mathfrak{E}(\mathcal{A}_1)$

Note that there are fewer factors here than in Table 2.2. To understand this, recall from the univariate cyclotomic polynomials that, for d odd, $\Phi_d(u^2) = \Phi_d(u)\Phi_{2d}(u)$. Thus for d odd one has:

$$\Phi_d(X_1, X_1^{-1}) = \Phi_d(X_1^{\frac{1}{2}}, X_1^{-\frac{1}{2}})\Phi_{2d}(X_1^{\frac{1}{2}}, X_1^{-\frac{1}{2}})$$

A priori, $\Phi_d(X_1^{\frac{1}{2}}, X_1^{-\frac{1}{2}}) \in \mathbb{Z}[X_1^{\frac{1}{2}}, X_1^{-\frac{1}{2}}]$, but more is true in fact:

Lemma 2.1.1. *For d odd, both $\Phi_d(X_1^{\frac{1}{2}}, X_1^{-\frac{1}{2}})$ and $\Phi_{2d}(X_1^{\frac{1}{2}}, X_1^{-\frac{1}{2}})$ are elements of $\mathbb{Z}[X_1, X_1^{-1}]$.*

Proof. Note that since the degree of Φ_d is even for $d \geq 3$, all terms of the homogenized polynomials $\Phi_d(X, Y)$ are of even total degree. In particular, each term of $\Phi_d(X_1^{\frac{1}{2}}, X_1^{-\frac{1}{2}})$ therefore has integer degree. The same argument applies for Φ_{2d} which proves the lemma. $\square$

Since the cyclotomic polynomials are irreducible, this completes the analysis of the factorizations of irreducible characters of $\mathcal{A}_1$. Before moving on, we note that the above analysis indicates that if $d|n$ then $\widehat{\chi}_{[d]}|\widehat{\chi}_{[n]}$ in $\mathfrak{E}(\mathcal{A}_1)$; this explains the appearance of the repeated factors in Table 2.1. This divisibility result will also follow from Theorem 3.1.1 which we prove in §3.1.1.

2.1.2 The Lie Group $\mathcal{A}_2$

The case of $\mathcal{A}_1$ having been completely handled, we now turn to the next simplest case: $SU(3) = \mathcal{A}_2$. It turns out that factorization properties observed in $\mathcal{A}_1$ are in some ways unique to $\mathcal{A}_1$; for example, if $G \neq \mathcal{A}_1$ then there are irreps which do not factor in $\mathfrak{R}(G)$ or $\mathfrak{E}(G)$. As we will see, $\mathcal{A}_2$ better displays typical behavior which appears in the higher rank Lie groups. In particular, $\mathcal{A}_2$ clearly displays factorization relationships which occur in all Lie Groups and will exhibit the typical factorization properties of the simply-laced Lie Groups. As we are only working in $\mathcal{A}_2$ thusfar, we again omit reference to the group throughout this section.

$\mathcal{A}_2$ has two fundamental weights ω_1 and ω_2, so irreducible representations have highest weights of the form $[n_1, n_2]$. Note that $\pi_{\omega_2} = \Lambda^2\pi_{\omega_1} = \pi^*_{\omega_1}$ and $\pi_{\omega_1} = \Lambda^2\pi_{\omega_2} = \pi^*_{\omega_2}$, so that all properties of the fundamental representations will be reflexive in the two indices. For example, the weights of $\pi_{\omega_1} = \pi_{[1,0]}$ are $[1, 0]$,

$[-1,1]$, and $[0,-1]$; by reflexivity the weights of π_{ω_2} are $[0,1]$, $[1,-1]$, $[-1,0]$. In this case, as elements of $\mathfrak{E}(\mathcal{A}_2)$ the fundamental characters are given by:

$$\chi_{\omega_1} = X_1 + X_1^{-1}X_2 + X_2^{-1}$$

$$\chi_{\omega_2} = X_2 + X_1X_2^{-1} + X_1^{-1}$$

The Weyl group of $\mathcal{A}_2$ is of order 6, and in particular the $E_{[n_1,n_2]}(\mathcal{A}_2)$ appearing in the Weyl Character Formula are given by:

$$\begin{aligned} E_{[n_1,n_2]} =& X_1^{n_1}X_2^{n_2} + X_1^{n_2}X_2^{-n_1-n_2} + X_1^{-n_1-n_2}X_2^{n_1} \\ & - X_1^{n_1+n_2}X_2^{-n_2} - X_1^{-n_1}X_2^{n_1+n_2} - X_1^{-n_2}X_2^{-n_1} \end{aligned}$$

Using Klimyk's Formula, basic properties of plethysms of the fundamental representations of $SU(3)$, and some rearranging, one can derive the following general recursive relationships among these polynomials in terms of symmetric and exterior powers:

$$\begin{aligned} \pi_{[n_1,n_2]} &= \left(\pi_{[n_1,0]} \otimes \pi_{[0,n_2]}\right) \ominus \left(\pi_{[n_1-1,0]} \otimes \pi_{[0,n_2-1]}\right) \\ \pi_{[n_1,0]} &= Sym^{n_1}(\pi_{[1,0]}) \\ &= \sum_{i=1}^{3}(-1)^{i-1}\Lambda^i(\pi_{[1,0]}) \otimes Sym^{n_1-i}(\pi_{[1,0]}) \\ &= \left(\pi_{[1,0]} \otimes Sym^{n_1-1}(\pi_{[1,0]})\right) \ominus \left(\pi_{[0,1]} \otimes Sym^{n_1-2}(\pi_{[1,0]})\right) \oplus Sym^{n_1-3}(\pi_{[1,0]}) \\ &= \left(\pi_{[1,0]} \otimes \pi_{[n_1-1,0]}\right) \ominus \left(\pi_{[0,1]} \otimes \pi_{[n_1-2,0]}\right) \oplus \pi_{[n_1-3,0]} \end{aligned}$$

$$
\begin{aligned}
\pi_{[0,n_2]} &= Sym^{n_2}(\pi_{[0,1]}) \\
&= \sum_{i=1}^{3}(-1)^{i-1}\Lambda^i(\pi_{[0,1]}) \otimes Sym^{n_2-i}(\pi_{[0,1]}) \\
&= \big(\pi_{[0,1]} \otimes Sym^{n_2-1}(\pi_{[0,1]})\big) \ominus \big(\pi_{[1,0]} \otimes Sym^{n_2-2}(\pi_{[0,1]})\big) \oplus Sym^{n_2-3}(\pi_{[0,1]}) \\
&= \big(\pi_{[0,1]} \otimes \pi_{[0,n_2-1]}\big) \ominus \big(\pi_{[1,0]} \otimes \pi_{[0,n_2-2]}\big) \oplus \pi_{[0,n_2-3]}
\end{aligned}
$$

Using these relationships, we can quickly generate the polynomials of irreps in $\mathfrak{R}(\mathcal{A}_2)$ without having to apply Algorithm 1.1.1 to each individual irrep. Doing so, we construct Table 2.4 showing the factorizability in $\mathfrak{R}(\mathcal{A}_2)$ of small index irreps.

Looking at this table, there are not many patterns readily apparent. Not surprisingly we see the table is symmetric across the main diagonal, which is to be expected due to the reflexivity of $\pi_{[1,0]}$ and $\pi_{[0,1]}$ noted earlier. To better understand the patterns arising in this table, let us examine a few of the factorizations themselves.

To begin with, in the second row, we have that the $\pi_{[3,1]}$, $\pi_{[5,1]}$, $\pi_{[7,1]}$, $\pi_{[9,1]}$, and $\pi_{[11,1]}$ representations all factor in $\mathfrak{R}(\mathcal{A}_2)$. Recalling the shorthand notation V_i from §1.1.1 so that $V_1 = \pi_{[1,0]}$ and $V_2 = \pi_{[0,1]}$, the factorizations of the first few are given by:

$$\pi_{[3,1]} = (V_1V_2 - 1)\cdot(V_1^2 - 2V_2)$$

$$\pi_{[5,1]} = (V_1V_2 - 1)\cdot(V_1^4 - 4V_1^2V_2 + 3V_2^2 + 2V_1)$$

$$\pi_{[7,1]} = (V_1V_2 - 1)\cdot(V_1^6 - 6V_1^4V_2 + 10V_1^2V_2^2 - 4V_2^3 + 4V_1^3 - 8V_1V_2 + 1)$$

$n_2 \backslash n_1$	0	1	2	3	4	5	6	7	8	9	10	11	$\dots$
0	-	-	-	-	-	-	-	-	-	-	-	-	$\dots$
1	-	-	-	X	-	X	-	X	-	X	-	X	$\dots$
2	-	-	-	-	-	X	-	-	X	-	-	X	$\dots$
3	-	X	-	X	-	X	-	X	-	X	-	X	$\dots$
4	-	-	-	-	-	-	-	-	-	X	-	-	$\dots$
5	-	X	X	X	-	X	-	X	X	X	-	X	$\dots$
6	-	-	-	-	-	-	-	-	-	-	-	-	$\dots$
7	-	X	-	X	-	X	-	X	-	X	-	X	$\dots$
8	-	-	X	-	-	X	-	-	X	-	-	X	$\dots$
9	-	X	-	X	X	X	-	X	-	X	-	X	$\dots$
10	-	-	-	-	-	-	-	-	-	-	-	-	$\dots$
11	-	X	X	X	-	X	-	X	X	X	-	X	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 2.4: X = Irreducible Representation $\pi_{[n_1,n_2]}$ factors in $\mathfrak{R}(\mathcal{A}_2)$

These irreps have the common factor V_1V_2-1, so we might guess that this factor is important. To test this hypothesis, we construct Table 2.5 showing which irreps are divisible by this factor.

$n_2 \backslash n_1$	0	1	2	3	4	5	6	7	8	9	10	11	...
0	-	-	-	-	-	-	-	-	-	-	-	-	...
1	-	X	-	X	-	X	-	X	-	X	-	X	...
2	-	-	-	-	-	-	-	-	-	-	-	-	...
3	-	X	-	X	-	X	-	X	-	X	-	X	...
4	-	-	-	-	-	-	-	-	-	-	-	-	...
5	-	X	-	X	-	X	-	X	-	X	-	X	...
6	-	-	-	-	-	-	-	-	-	-	-	-	...
7	-	X	-	X	-	X	-	X	-	X	-	X	...
8	-	-	-	-	-	-	-	-	-	-	-	-	...
9	-	X	-	X	-	X	-	X	-	X	-	X	...
10	-	-	-	-	-	-	-	-	-	-	-	-	...
11	-	X	-	X	-	X	-	X	-	X	-	X	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 2.5: X = $(V_1V_2 - 1)$ divides $\pi_{[n_1,n_2]}$ in $\mathfrak{R}(\mathcal{A}_2)$

Table 2.5 highlights two important facts about the common factor $(V_1V_2 - 1)$. Firstly, it divides a very regular pattern of character polynomials in $\mathfrak{R}(\mathcal{A}_2)$; namely, it divides $\pi_{[n_1,n_2]}$ whenever both n_1 and n_2 are odd (later we will prove this holds in general). Secondly, in Table 2.5 there is now an X in the $(1,1)$-position whereas in Table 2.4 there was no X in this position.

This latter fact is not surprising once we calculate the image of $\pi_{[1,1]}$ in $\mathfrak{R}(\mathcal{A}_2)$ which turns out to be $V_1V_2 - 1$. Thus the common factor arising in the second row and elsewhere in the table is exactly $\pi_{[1,1]}$. In light of this, the lack of an X in Table 2.4 in the $(1,1)$-position is not surprising; we can view this representation as factorizing as $(V_1V_2 - 1) \cdot 1$ in order to 'fill in' the missing factorization in the $(1,1)$-position of Table 2.4.

We have now explained some of the factorizations in Table 2.4, but not all of them, so we move on to the third row where we find that $\pi_{[5,2]}$, $\pi_{[8,2]}$, and $\pi_{[11,2]}$ factor as follows:

$$\begin{aligned}
\pi_{[5,2]} &= (V_1^2V_2^2 - V_1^3 - V_2^3) \cdot (V_1^3 - 3V_1V_2 + 3) \\
\pi_{[8,2]} &= (V_1^2V_2^2 - V_1^3 - V_2^3) \\
&\qquad \cdot (V_1^6 - 6V_1^4V_2 + 9V_1^2V_2^2 - V_2^3 + 6V_1^3 - 15V_1V_2 + 6) \\
\pi_{[11,2]} &= (V_1^2V_2^2 - V_1^3 - V_2^3) \cdot \\
&\qquad (V_1^9 - 9V_1^7V_2 + 27V_1^5V_2^2 - 29V_1^3V_2^3 + 6V_1V_2^4 + 9V_1^6 - 48V_1^4V_2 \\
&\qquad + 63V_1^2V_2^2 - 6V_2^3 + 21V_1^3 - 45V_1V_2 + 10)
\end{aligned}$$

Again we see that each of these polynomials has a common factor, in this case $(V_1^2V_2^2 - V_1^3 - V_2^3)$. Checking when this factor contributes to Table 2.4, we find it does exactly when:

$$n_1 \equiv n_2 \equiv (2 \mod 3)$$

A short computation shows that this factor is the image of $\pi_{[2,2]}$ in $\mathfrak{R}(\mathcal{A}_2)$; note the similarity between the common factors in the second and third rows of Table 2.4. We thus begin to understand the apparent chaos of Table 2.4 as a superposition of many regular patterns; based on our observations a reasonable first guess is that each pattern is related to a factor of the form $\pi_{[n,n]}$.

Before proceeding further, since we understand the common factors which arise, we would also like to understand the cofactors as well if possible. We begin by returning to the second row of Table 2.4 where we had the following cofactors of $\pi_{[1,1]}$:

$$
\begin{aligned}
cof_{[3,1]} &:= \left(\frac{\pi_{[3,1]}}{\pi_{[1,1]}}\right) = V_1^2 - 2V_2 \\
cof_{[5,1]} &:= \left(\frac{\pi_{[5,1]}}{\pi_{[1,1]}}\right) = V_1^4 - 4V_1^2V_2 + 3V_2^2 + 2V_1 \\
cof_{[7,1]} &:= \left(\frac{\pi_{[7,1]}}{\pi_{[1,1]}}\right) = V_1^6 - 6V_1^4V_2 + 10V_1^2V_2^2 - 4V_2^3 + 4V_1^3 - 8V_1V_2 + 1
\end{aligned}
$$

At first glance, there is very little we can say about these cofactors; a quick calculation of their decomposition into irreducibles shows none of them are honest representations of $\mathcal{A}_2$. To better understand exactly which virtual representations they are, we first look at their dimensions. Since both V_1 and V_2 are 3-dimensional, the dimension of an arbitrary element of $\mathfrak{R}(\mathcal{A}_2)$ is found by specializing $V_1 = V_2 = 3$. Upon doing so, we have $\dim(cof_{[3,1]}) = 3$, $\dim(cof_{[5,1]}) = 6$ and $\dim(cof_{[7,1]}) = 10$. These dimensions are irreducible dimensions of $\mathcal{A}_2$-modules, but we have already seen the cofactors are not characters of any irreps. Thus we look deeper to find out what, if any, is the connection to the irreps of $\mathcal{A}_2$.

Cofactor	Dimension	Image in $\mathfrak{R}(\mathcal{A}_2)$	Image in $\mathfrak{E}(\mathcal{A}_2)$
$cof_{[3,1]}$	3	$V_1^2 - 2V_2$	$X_1^2 + X_1^{-2}X_2^2 + X_2^{-2}$
$cof_{[5,1]}$	6	$V_1^4 - 4V_1^2V_2 + 3V_2^2 + 2V_1$	$X_1^4 + X_1^{-4}X_2^4 + X_2^{-4}$ $+X_2^2 + X_1^2X_2^{-2} + X_1^{-2}$
$cof_{[7,1]}$	10	$V_1^6 - 6V_1^4V_2 + 10V_1^2V_2^2$ $-4V_2^3 + 4V_1^3 - 8V_1V_2 + 1$	$X_1^6 + X_1^{-6}X_2^6 + X_2^{-6}$ $+X_1^2X_2^2 + X_1^{-4}X_2^2 + X_1^2X_2^{-4}$ $X_1^{-2}X_2^4 + X_1^4X_2^{-2} + X_1^{-2}X_2^{-2} + 1$

Table 2.6: Images in $\mathfrak{E}(\mathcal{A}_2)$ of low-dimensional cofactors of $\pi_{[1,1]}$

Thusfar we have worked only in $\mathfrak{R}(\mathcal{A}_2)$ to find out properties of these cofactors. Having said about as much about them as we can in $\mathfrak{R}(\mathcal{A}_2)$ for now, the logical next step is to look at their images in $\mathfrak{E}(\mathcal{A}_2)$ and see what can be said there. Doing so, gives Table 2.6.

Viewed in $\mathfrak{E}(\mathcal{A}_2)$, these cofactors assume much more regularity. For example, we notice that each of the individual monomials of the images of the cofactors has exponents divisible by 2. Since each term is a perfect square, we may therefore take its square root and examine the character that results. Doing so we find that the resulting characters are exactly the characters $\chi_{[1,0]}$, $\chi_{[2,0]}$ and $\chi_{[3,0]}$ respectively. Thus these three cofactors are $\psi^2\chi_{[1,0]}$, $\psi^2\chi_{[2,0]}$ and $\psi^2\chi_{[3,0]}$ respectively.

We can therefore summarize our the factorizations in the second row of Table

2.4 as follows:

$$\chi_{[3,1]} = \chi_{[1,1]} \cdot \psi^2 \chi_{[1,0]}$$

$$\chi_{[5,1]} = \chi_{[1,1]} \cdot \psi^2 \chi_{[2,0]}$$

$$\chi_{[7,1]} = \chi_{[1,1]} \cdot \psi^2 \chi_{[3,0]}$$

Likewise the cofactors of $\pi_{[2,2]}$ in the the third row of Table 2.4, when viewed in $\mathfrak{E}(\mathcal{A}_2)$, involve ψ^3:

$$\chi_{[5,2]} = \chi_{[2,2]} \cdot \psi^3 \chi_{[1,0]}$$

$$\chi_{[8,2]} = \chi_{[2,2]} \cdot \psi^3 \chi_{[2,0]}$$

$$\chi_{[11,2]} = \chi_{[2,2]} \cdot \psi^3 \chi_{[3,0]}$$

Observing that the irreps in the second row of Table 2.4 which factor are those for which both n_1 and n_2 are odd while and in the third row those that factor are those for which both n_1 and n_2 are congruent to $2 \mod 3$, we can begin to guess a general pattern. First define N, and k_i for $i = 1, 2$ by:

$$N := \gcd(n_1, n_2)$$

$$k_i := \tfrac{n_i}{N}$$

Then the observed patterns may be summarized as follows:

$$\widehat{\chi}_{[n_1,n_2]} = \widehat{\chi}_{[N,N]} \cdot \psi^N \widehat{\chi}_{[k_1,k_2]} \tag{2.1.1}$$

$$\widehat{\pi}_{[n_1,n_2]} = \widehat{\pi}_{[N,N]} \cdot \Psi^N \widehat{\pi}_{[k_1,k_2]} \tag{2.1.2}$$

Remark. In the case that $N = 1$, this statement is vacuously true since on the RHS we have $\widehat{\chi}_{[1,1]} = 1$ while the second factor is $\psi^1 \widehat{\chi}_{[n_1,n_2]} = \widehat{\chi}_{[n_1,n_2]}$.

It is not hard to check that the positions of all factorizations appearing in Table 2.4 can be accounted for by Formula 2.1.1. Such factorizations are what we call 'ordinary' factorizations and correspond to the ordinary Adams' operations. Since the Adams' operations are defined for all G, it is plausible to assume similar factorizations occur in general, and indeed in §3.1.1 we will prove Theorem 3.1.1 which shows that this is indeed the case.

For some representations, including all irreps along the main diagonal which factor in Table 2.4, Formula 2.1.1 predicts multiple possible factorizations. In §3.2 we will handle this situation when we introduce the notion of Γ-factorizations and prove their existence in the irreps of all Lie Groups.

2.2 Nonsimply-Laced Groups

Having seen numerous factorization properties in $\mathcal{A}_1$ and $\mathcal{A}_2$, and in particular how many factorizations in $\mathfrak{R}(\mathcal{A}_2)$ are related to the Adams' operations, we proceed to examine the nonsimply-laced groups. As we will quickly discover, the nonsimply-laced groups behave slightly differently from the simply-laced groups, and thus we will examine each of them in turn, beginning with the exceptionals $\mathcal{G}_2$ and $\mathcal{F}_4$. Although one would assume that $\mathcal{B}_2$ would be easiest to deal with, we will hold off on its analysis until after these two cases due to special considerations which arise in the $\mathcal{B}_n$ and $\mathcal{C}_n$ series.

Although we have not yet proven that the ordinary factorizations seen in $\mathfrak{R}(\mathcal{A}_2)$ occur in general, as noted before, Theorem 3.1.1 will show they do exist and thus will guarantees a similar factorization pattern to that of $\mathcal{A}_2$ among the irreducible representations in $\mathfrak{R}(G)$ for the nonsimply-laced groups we study below. Thus, in these discussions we will assume Theorem 3.1.1 and focus our attention exclusively on factorizations not predicted by Theorem 3.1.1.

As we now will be working in several groups, several of the same rank to one another, we will no longer omit references to the group in our notation for irreps, characters, etc.

2.2.1 The Lie Group $\mathcal{G}_2$

Proceeding as with $\mathcal{A}_2$, $\mathfrak{R}(\mathcal{G}_2)$ is generated by two fundamental representations; however as the simple roots of $\mathcal{G}_2$ are fundamentally different, the dimensions of these fundamental representations differ; in our convention ω_1 corresponds to the short simple root, hence $\pi_{\omega_1}(\mathcal{G}_2)$ is the natural 7-dimensional representation of $\mathcal{G}_2$ acting as automorphisms of the Cayley numbers. Likewise, $\pi_{\omega_2}(\mathcal{G}_2)$ is the adjoint representation of dimension 14.

We therefore proceed to calculate factorizations in $\mathfrak{R}(\mathcal{G}_2)$ using a combination of Algorithm 1.1.1 and Klimyk's Formula; this leads to Table 2.7.

$n_2 \backslash n_1$	0	1	2	3	4	5	6	7	8	9	10	11	$\dots$
0	-	-	-	-	-	X	-	-	X	-	-	X	$\dots$
1	-	X	X	X	-	X	-	X	X	X	-	X	$\dots$
2	-	-	X	-	-	X	-	-	X	-	-	X	$\dots$
3	-	X	X	X	-	X	-	X	X	X	-	X	$\dots$
4	-	-	X	-	X	X	-	-	X	X	-	X	$\dots$
5	-	X	X	X	-	X	-	X	X	X	-	X	$\dots$
6	-	-	X	-	-	X	X	-	X	-	-	X	$\dots$
7	-	X	X	X	-	X	-	X	X	X	-	X	$\dots$
8	-	-	X	-	-	X	-	-	X	-	-	X	$\dots$
9	-	X	X	X	X	X	-	X	X	X	-	X	$\dots$
10	-	-	X	-	-	X	-	-	X	-	X	X	$\dots$
11	-	X	X	X	-	X	-	X	X	X	-	X	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 2.7: X = $\pi_{[n_1,n_2]}(\mathcal{G}_2)$ factors in $\mathfrak{R}(\mathcal{G}_2)$

Here we see a very different factorization pattern than that seen in Table 2.4. If we ignore the factorizations predicted by Theorem 3.1.1, then we obtain Table 2.8.

Unlike Table 2.5 in $\mathcal{A}_2$, the patterns of Table 2.8 are somewhat easier to see, although as before they are not entirely predictable yet. In particular, we see that the new factorizations occur along the main diagonal and in certain columns, but it

$n_2 \backslash n_1$	0	1	2	3	4	5	6	7	8	9	10	11	…
0	-	-	-	-	-	X	-	-	X	-	-	X	…
1	-	X	X	-	-	-	-	-	X	-	-	-	…
2	-	-	X	-	-	-	-	-	-	-	-	-	…
3	-	-	X	-	-	-	-	-	X	-	-	-	…
4	-	-	X	-	X	X	-	-	X	-	-	X	…
5	-	-	-	-	-	-	-	-	-	-	-	-	…
6	-	-	X	-	-	X	X	-	X	-	-	X	…
7	-	-	X	-	-	-	-	-	X	-	-	-	…
8	-	-	-	-	-	-	-	-	-	-	-	-	…
9	-	-	X	-	-	-	-	-	X	-	-	-	…
10	-	-	X	-	-	X	-	-	X	-	X	X	…
11	-	-	-	-	-	-	-	-	-	-	-	-	…
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 2.8: X $= \pi_{[n_1,n_2]}(\mathcal{G}_2)$ factors in $\mathfrak{R}(\mathcal{G}_2)$, but not predicted by Theorem 3.1.1

is not immediately clear as to which entries along the diagonal or in these columns correspond to new factorizations.

We defer dealing with the new factorizations along the main diagonal of Table 2.8 until §3.2. To understand the new patterns in the columns, we proceed as before to look for common factors and then determine the cofactors. Thus we obtain:

$$\pi_{[2,1]}(\mathcal{G}_2) = (V_1^2 - V_1 - V_2 - 1) \cdot (V_2 - V_1)$$

$$\pi_{[2,2]}(\mathcal{G}_2) = (V_1^2 - V_1 - V_2 - 1) \cdot (V_2^2 - V_1^3 + V_1V_2 + V_1^2 + V_2 + 2V_1 - 1)$$

$$\begin{aligned}\pi_{[2,3]}(\mathcal{G}_2) = (V_1^2 - V_1 - V_2 - 1)\cdot \\ (V_2^3 - 2V_1^3V_2 + 2V_1^4 + 3V_1V_2^2 - 3V_1^2V_2 + 3V_2^2 - 2V_2^3 + 3V_1V_2 \\ - 3V_1^2 + V_2 + 2V_1)\end{aligned}$$

$$\pi_{[5,0]}(\mathcal{G}_2) = (V_1^2 - V_1 - V_2 - 1) \cdot (V_1^3 - 3V_1V_2 - V_1 - 2V_2)$$

As with the factorizations in $\mathfrak{R}(\mathcal{A}_2)$, we have a common factor and further calculation shows that this factor divides the other non-diagonal representations in Table 2.8. In fact, checking further, one finds that this factor is equal to $\pi_{[2,0]}(\mathcal{G}_2)$ and divides $\pi_{[n_1,n_2]}(\mathcal{G}_2)$ whenever n_1 = 2,5,8, or 11 in Table 2.7. We therefore set out as before to determine the cofactors of these representations and in particular to determine the underlying patterns. We first summarize the dimensions of these cofactors in Table 2.9.

Comparing the dimensions of cofactors in Table 2.9 to dimensions of irreps of $\mathcal{G}_2$ (Table 2.10) one finds that the dimensions of the cofactors are exactly dimensions of irreducible $\mathcal{G}_2$-modules. Thus, as in §2.1.2 we look for an explanation for these common dimensions; to find this connection we will proceed as before to look at the images of the cofactors in $\mathfrak{E}(\mathcal{G}_2)$.

$n_2 \backslash n_1$	0	1	2	3	4	5	6	7	8	9	10	11	$\ldots$
0	-	-	1	-	-	14	-	-	77	-	-	273	$\ldots$
1	-	-	7	-	-	64	-	-	286	-	-	896	$\ldots$
2	-	-	27	-	-	189	-	-	729	-	-	2079	$\ldots$
3	-	-	77	-	-	448	-	-	1547	-	-	4096	$\ldots$
4	-	-	182	-	-	924	-	-	2926	-	-	7293	$\ldots$
5	-	-	378	-	-	1728	-	-	5103	-	-	12096	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 2.9: Dimensions of cofactors of $\pi_{[2,0]}(\mathcal{G}_2)$ in $\mathfrak{R}(\mathcal{G}_2)$

$n_2 \backslash n_1$	0	1	2	3	4	5	$\ldots$
0	1	7	27	77	182	378	$\ldots$
1	14	64	189	448	924	1728	$\ldots$
2	77	286	729	1547	2926	5103	$\ldots$
3	273	896	2079	4096	7293	12096	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 2.10: Dimensions of $\pi_{[n_1,n_2]}(\mathcal{G}_2)$

The fundamental characters of $\mathcal{G}_2$ are:

$$\chi_{\omega_1}(\mathcal{G}_2) = X_1 + X_1^{-1}X_2 + X_1^2X_2^{-1} + 1 + X_1^{-2}X_2 + X_1X_2^{-1} + X_1^{-1}$$

$$\begin{aligned}\chi_{\omega_2}(\mathcal{G}_2) = {} & X_2 + X_1^3X_2^{-1} + X_1 + X_1^{-1}X_2 + X_1^2X_2^{-1} + X_1^{-3}X_2^2 + 2 \\ & + X_1^3X_2^{-2} + X_1^{-2}X_2 + X_1X_2^{-1} + X_1^{-1} + X_1^{-3}X_2 + X_2^{-1}\end{aligned}$$

Comparing the 7-dimensional cofactor $cof(\chi_{[2,1]}(\mathcal{G}_2)) := \frac{\chi_{[2,1]}}{\chi_{[2,0]}}$ with $\chi_{\omega_1}(\mathcal{G}_2)$ one has:

$$cof(\chi_{[2,1]}) = X_2 + X_1^3 X_2^{-1} + X_1^{-3} X_2^2 + 1 + X_1^3 X_2^{-2} + X_1^{-3} X_2 + X_2^{-1}$$

$$\chi_{\omega_1} = X_1 + X_1^{-1} X_2 + X_1^2 X_2^{-1} + 1 + X_1^{-2} X_2 + X_1 X_2^{-1} + X_1^{-1}$$

After some inspection, one sees that χ_{ω_1} and $cof(\chi_{[2,1]})$ may be related to one another by any of several possible maps. The easiest of these to describe is given by:

$$X_1 \mapsto X_2$$

$$X_2 \mapsto X_1^3$$

We denote this map ξ. Upon applying ξ to other irreducible characters, we find that they too coincide with the the other cofactors. ξ applied to a character is not easily describable in terms of an action on $\mathcal{G}_2$ like the Adams operations were, but it is easy to check that, as endomorphisms of $\mathfrak{C}(\mathcal{G}_2)$ it is related to the Adams operations by:

$$\xi \circ \xi = \psi^3$$

ξ thus satisfies relationship 1.1.2 and is in fact related to the exotic map $\varepsilon : B\mathcal{G}_2 \rightarrow B\mathcal{G}_2$ introduced in §1.1.2. Clearly the induced action on weights is given by:

$$\epsilon[n_1, n_2] = [3n_2, n_1]$$

Finally, by brute force computation, one finds that the corresponding endomorphism Ξ of $\mathfrak{R}(\mathcal{G}_2)$ is given by:

$$\Xi(V_1) = V_2 - V_1$$

$$\Xi(V_2) = V_1^3 - 3V_1V_2 - V_1 - 2V_2$$

Returning to the information gleaned from Tables 2.9 and 2.10 we can therefore summarize the new factorizations as follows:

$$\widehat{\pi}_{[3n_2,n_1]}(\mathcal{G}_2) = \widehat{\pi}_{[3,1]}(\mathcal{G}_2) \cdot \Xi\widehat{\pi}_{[n_1,n_2]}(\mathcal{G}_2) \tag{2.2.1}$$

Note the similarity between (2.2.1) and the ordinary factorizations in $\mathfrak{R}(\mathcal{A}_2)$ given by Formula 2.1.1; the major changes are that the indices n_1 and n_2 switch spots between the LHS and the second factor on the RHS and the first factor on the RHS is no longer of the form $\widehat{\chi}_{m[\rho]}$. In light of what we know about the map ϵ on weights of $\mathcal{G}_2$ and the form of the ordinary factorizations with respect to ψ^m we rewrite (2.2.1) alongside (2.1.1):

$$\widehat{\pi}_{m[n_1,n_2]}(\mathcal{G}_2) = \widehat{\pi}_{m[\rho]}(\mathcal{G}_2) \cdot \Psi^m\widehat{\pi}_{[n_1,n_2]}(\mathcal{G}_2)$$

$$\widehat{\pi}_{\epsilon[n_1,n_2]}(\mathcal{G}_2) = \widehat{\pi}_{\epsilon[\rho]}(\mathcal{G}_2) \cdot \Xi\widehat{\pi}_{[n_1,n_2]}(\mathcal{G}_2)$$

Both factorizations have now taken essentially the same form, showing that even though the exotic map is quite different from the ordianry Adams' maps, nevertheless both give rise to very similar looking series of factorizations. As the factorizations in (2.2.1) are related to the exotic map described by Adams, we call them exotic factorizations in analogy with the ordinary factorizations which were related to ordinary Adams' operations.

2.2.2 The Lie Group $\mathcal{F}_4$

When we go to examine factorizations in $\mathfrak{R}(\mathcal{F}_4)$, we are confronted with many technical limitations. Whereas the Weyl Groups of $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{G}_2$ were small (of orders 2, 6 and 12 respectively), the Weyl Group of $\mathcal{F}_4$ is comparatively large (of order 1152) which makes actual computations of the characters via the Weyl Character Formula vastly more difficult in terms of time and computing power required. Once we have the characters we would then need to do further calculations to find their images in $\mathfrak{R}(\mathcal{F}_4)$. Once this is done, the lattice of irreducible representations is still 4-dimensional, so to plot which representations factor we would have to look at 2- or 3-dimensional slices of this lattice and try to reconstruct the entire picture based on these slices.

Because of these obstructions, we would first like to narrow down our search in order to locate irreducible representations of $\mathcal{F}_4$ which are likely to factor in $\mathfrak{R}(\mathcal{F}_4)$ and then to examine these in order to verify our predictions. In order to find factorizations in $\mathfrak{R}(\mathcal{F}_4)$ not covered by Theorem 3.1.1, we should begin our search by looking at the exotic map $\xi_{\mathcal{F}_4} : \mathfrak{E}(\mathcal{F}_4) \to \mathfrak{E}(\mathcal{F}_4)$ which by Theorem 1.1.2 has some similar properties to those of the exotic map $\xi_{\mathcal{G}_2}$.

We now examine how to construct $\xi_{\mathcal{F}_4}$ given only that its action on $\mathfrak{E}(\mathcal{F}_4)$ is induced from the admissible map $\varepsilon_{\mathcal{F}_4}$ whose action also has not yet been explicitly specified. To do so, we will first take a closer look at the map $\xi_{\mathcal{G}_2}$.

Recalling that $\xi_{\mathcal{G}_2}$ induced a map $\epsilon_{\mathcal{G}_2}$ on the weight lattice of $\mathcal{G}_2$, we begin by looking more closely at the action of $\epsilon_{\mathcal{G}_2}$. We know how $\epsilon_{\mathcal{G}_2}$ acts on the fundamental weights of $\mathcal{G}_2$ which are dual to the simple roots, we therefore know the action of $\varepsilon_{\mathcal{G}_2}$ on the simple roots of $\mathcal{G}_2$. Now using what we know about $\epsilon_{\mathcal{G}_2}$, it is not hard to see that the action $\varepsilon_{\mathcal{G}_2}$ on the root lattice of $\mathcal{G}_2$ thus maps the short simple root α_1 to α_2 and maps the long simple root α_2 to $3\alpha_1$.

In light of this, the appearance of ψ^3 in relation to $\xi_{\mathcal{G}_2}$ is no accident either. As the length of α_2 is $\sqrt{3}$ times the length of α_1, geometrically the induced action of $\epsilon_{\mathcal{G}_2}$ reflects the root lattice over a mirror while simultaneously stretching it by a factor of $\sqrt{3}$ in all directions; clearly applying such an operation twice results in simply stretching the entire lattice by a factor of 3 which is the induced action of ψ^3. The induced action on the root lattice is exactly the admissible map $\varepsilon_{\mathcal{G}_2}$ introduced in Theorem 1.1.2.

In the root lattice of $\mathcal{F}_4$, we have two short simple roots (α_1 and α_2) and two long simple roots (α_3 and α_4), and the long roots are $\sqrt{2}$ times the short roots in length. At first it might seem that this gives us several possible choices for the action of $\varepsilon_{\mathcal{F}_4}$ on the root lattice; but if we also want to preserve angles between roots then it is not hard to see that there is a unique map satisfying these conditions, given by:

$$\alpha_1 \mapsto \alpha_4$$

$$\alpha_2 \mapsto \alpha_3$$

$$\alpha_3 \mapsto 2\alpha_2$$

$$\alpha_4 \mapsto 2\alpha_1$$

Hence we find that the map $\epsilon_{\mathcal{F}_4}$ on the weights of $\mathcal{F}_4$ and the map $\xi_{\mathcal{F}_4} : \mathfrak{E}(\mathcal{F}_4) \to \mathfrak{E}(\mathcal{F}_4)$ are given by:

ω_i	$\epsilon_{\mathcal{F}_4}\omega_i$
ω_1	ω_4
ω_2	ω_3
ω_3	$2\omega_2$
ω_4	$2\omega_1$

Generator X_i	$\xi_{\mathcal{F}_4}X_i$
X_1	X_4
X_2	X_3
X_3	X_2^2
X_4	X_1^2

Remark. It is clear from this action that $\xi_{\mathcal{F}_4}$ satisfies $\xi_{\mathcal{F}_4} \circ \xi_{\mathcal{F}_4} = \psi^2_{\mathcal{F}_4}$ in analogy with our the observation that $\xi_{\mathcal{G}_2} \circ \xi_{\mathcal{G}_2} = \psi^3_{\mathcal{G}_2}$ and also in agreement with (1.1.2). As we are now working exclusively in $\mathcal{F}_4$ we will drop the subscript $\mathcal{F}_4$'s on our maps for the remainder of this section.

Modifying 2.2.1 so as to use this new map throughout, we therefore predict factorizations in $\mathfrak{R}(\mathcal{F}_4)$ of the form:

$$\widehat{\pi}_{[2n_4,2n_3,n_2,n_1]}(\mathcal{F}_4) = \widehat{\pi}_{[2,2,1,1]}(\mathcal{F}_4) \cdot \Xi\widehat{\pi}_{[n_1,n_2,n_3,n_4]}(\mathcal{F}_4) \tag{2.2.2}$$

In particular we have predicted that the irreducible representation $\widehat{\pi}_{[2,2,1,1]}(\mathcal{F}_4) = \pi_{[1,1,0,0]}(\mathcal{F}_4)$ of dimension 4096 will appear as a common factor in these factorizations. Considering the case of minimal dimension, $\widehat{\pi}_{[2,1,1,1]}(\mathcal{F}_4) = \pi_{[1,0,0,0]}(\mathcal{F}_4)$ of dimension 26, we therefore predict the smallest factorization in $\mathfrak{R}(\mathcal{F}_4)$ corresponding to ξ is:

$$\widehat{\pi}_{[2,2,1,2]}(\mathcal{F}_4) = \widehat{\pi}_{[2,2,1,1]}(\mathcal{F}_4) \cdot \Xi\widehat{\pi}_{[2,1,1,1]}(\mathcal{F}_4)$$

A quick check of dimensions shows that $\widehat{\pi}_{[2,2,1,2]}(\mathcal{F}_4) = \pi_{[1,1,0,1]}(\mathcal{F}_4)$ has dimension $106496 = 4096 \cdot 26$ so our prediction is at least consistent on the level of dimensions. With the aid of the LiE online computation package [LiE], we compute the image of $\pi_{[1,1,0,1]}(\mathcal{F}_4)$ in $\mathfrak{R}(\mathcal{F}_4)$ to be:

$$\begin{aligned}\pi_{[1,1,0,1]}(\mathcal{F}_4) &= V_1V_2V_4 - V_1V_2 - V_3V_4 + V_3V_1 - V_1V_4^2 + V_1^3 + V_2V_4 - V_1V_2 + V_1V_4 \\ &\quad - V_1^2 + V_4 - V_1 \\ &= (V_1V_2 - V_3 - V_1V_4 - V_1^2 + V_2 + V_1 + 1)(V_4 - V_1)\end{aligned}$$

Note that the LiE convention for $\mathcal{F}_4$ is the reverse of ours, so that for example what we would refer to as $\pi_{[1,2,3,4]}(\mathcal{F}_4)$ would be input in LiE as $\pi_{[4,3,2,1]}(\mathcal{F}_4)$.

Further calculation with LiE shows that $V_1V_2 - V_3 - V_1V_4 - V_1^2 + V_2 + V_1 + 1$ is indeed the image of $\pi_{[1,1,0,0]}(\mathcal{F}_4)$ in $\mathfrak{R}(\mathcal{F}_4)$. As for the second factor, let us examine its weights; by our predictions it should have the same weights as $\xi\chi_{[1,0,0,0]}(\mathcal{F}_4)$.

Weights of $\pi_{[1,0,0,0]}(\mathcal{F}_4)$					
$[1,0,0,0]$	$[-1,1,0,0]$	$[0,-1,1,0]$	$[0,1,-1,1]$	$[1,-1,0,1]$	$[0,1,0,-1]$
$[-1,0,0,1]$	$[1,-1,1,-1]$	$[-1,0,1,-1]$	$[1,1,-1,0]$	$[2,-1,0,0]$	$[-1,2,-1,0]$
$[1,-2,1,0]$	$[-2,1,0,0]$	$[-1,-1,1,0]$	$[1,0,-1,1]$	$[-1,1,-1,1]$	$[1,0,0,-1]$
$[0,-1,0,1]$	$[-1,1,0,-1]$	$[0,-1,1,-1]$	$[0,1,-1,0]$	$[1,-1,0,0]$	$[-1,0,0,0]$
$2\times[0,0,0,0]$					

Weights of $\pi_{[0,0,0,1]}(\mathcal{F}_4)\ominus\pi_{[1,0,0,0]}(\mathcal{F}_4)$					
$[0,0,0,1]$	$[0,0,1,-1]$	$[0,2,-1,0]$	$[2-2,1,0]$	$[2,0,-1,1]$	$[-2,0,1,0]$
$[2,0,0,-1]$	$[-2,2,-1,1]$	$[-2,2,0,-1]$	$[0,-2,1,1]$	$[0,0,-1,2]$	$[0,-2,2,-1]$
$[0,2,-2,1]$	$[0,0,1,-2]$	$[0,2,-1,-1]$	$[2,-2,0,1]$	$[2,-2,1,-1]$	$[-2,0,0,1]$
$[2,0,-1,0]$	$[-2,0,1,-1]$	$[-2,2,-1,0]$	$[0,-2,1,0]$	$[0,0,-1,1]$	$[0,0,0,-1]$
$2\times[0,0,0,0]$					

Table 2.11: Weights of Virtual Representations of $\mathcal{F}_4$

Calculation of the weights of $\pi_{[1,0,0,0]}(\mathcal{F}_4)$ and $V_4 - V_1 = \pi_{[0,0,0,1]}(\mathcal{F}_4)\ominus\pi_{[1,0,0,0]}(\mathcal{F}_4)$ gives Table 2.11.

It is easily checked that ϵ applied to any weight of $\pi_{[1,0,0,0]}(\mathcal{F}_4)$ gives a weight of $\pi_{[0,0,0,1]}(\mathcal{F}_4)\ominus\pi_{[1,0,0,0]}(\mathcal{F}_4)$ and conversely that all weights of $\pi_{[0,0,0,1]}(\mathcal{F}_4)\ominus\pi_{[1,0,0,0]}(\mathcal{F}_4)$ are obtained in this way, hence we have the desired result:

$$\xi\chi_{[1,0,0,0]}(\mathcal{F}_4) = \chi_{[0,0,0,1]}(\mathcal{F}_4) - \chi_{[1,0,0,0]}(\mathcal{F}_4)$$

Thus, based on our observation of the structure of the exotic factorizations in $\mathcal{G}_2$, we have been able to correctly predict exactly how $\pi_{[1,1,0,1]}(\mathcal{F}_4)$ factors in $\mathfrak{R}(\mathcal{F}_4)$. In §2.2.3 we will extend these predictions to $\mathcal{B}_n$ and $\mathcal{C}_n$ and in § 3.1.1 we will finally prove the general result (Theorem 3.1.2) about exotic factorizations in $\mathfrak{R}(G)$ when G is nonsimply-laced.

2.2.3 The Lie Groups $\mathcal{B}_n$ and $\mathcal{C}_n$

We now examine the series $\mathcal{B}_n$ and $\mathcal{C}_n$, starting with $\mathcal{B}_2 = \mathcal{C}_2$. For purposes of indexing weights and simple roots, we treat this case exclusively as $\mathcal{B}_2$.

$\mathcal{B}_2$ is small enough to effectively employ Klimyk's Formula and quickly generate irrep polynomials in $\mathfrak{R}(\mathcal{B}_2)$ and then check for factorizations as in $\mathcal{G}_2$. On the other hand, it is more instructive to examine $\mathcal{B}_2$ along the lines of our analysis in $\mathcal{F}_4$. Doing so we will be able to obtain predictions of which irreps of $\mathcal{B}_2$ we should expect to exhibit exotic factorizations and what those factors should look like.

From our work in $\mathcal{G}_2$ and subsequent predictions which turned out to hold for a relatively small example in $\mathcal{F}_4$, we first note that $\mathcal{B}_2$ has one long simple root, α_1 in our convention, and one short simple root α_2. The length of α_1 is $\sqrt{2}$ times the length of α_2. Based on what we have observed in $\mathcal{G}_2$ and $\mathcal{F}_4$, we therefore expect the action of $\varepsilon_{\mathcal{B}_2}$ on $\widetilde{\mathbb{T}}(\mathcal{B}_2)$ to be given by:

$$\alpha_1 \mapsto 2\alpha_2$$

$$\alpha_2 \mapsto \alpha_1$$

Therefore $\epsilon_{\mathcal{B}_2}$ and $\xi_{\mathcal{B}_2}$ should be given by:

ω_i	$\varepsilon_{\mathcal{B}_2}\omega_i$
ω_1	$2\omega_2$
ω_2	ω_1

Generator X_i	$\xi_{\mathcal{B}_2}X_i$
X_1	X_2^2
X_2	X_1

In particular, one predicts that the analogue of equations (2.2.1) and (2.2.2) for exotic factorizations in $\mathfrak{R}(\mathcal{B}_2)$ is given by:

$$\widehat{\pi}_{[n_2,2n_1]}(\mathcal{B}_2) = \widehat{\pi}_{[1,2]}(\mathcal{B}_2) \cdot \Xi_{\mathcal{B}_2}\widehat{\pi}_{[n_1,n_2]}(\mathcal{B}_2) \tag{2.2.3}$$

The common factor this time is $\pi_{\omega_2}(\mathcal{B}_2)$ which is the 4-dimensional spinor representation of $\mathcal{B}_2$; after unshifting the weights (2.2.3) therefore predicts that the spinor representation divides every representation $\pi_{[n_1,n_2]}(\mathcal{B}_2)$ such that n_2 is odd. To test this hypothesis, we proceed to calculate the polynomials of low-dimensional irreps in $\mathfrak{R}(\mathcal{B}_2)$ and find out when they factor. The result is Table 2.12.

As with $\mathcal{G}_2$, one sees that this table of factorizations contains the factorizations predicted by Theorem 3.1.1, factorizations down the main diagonal not predicted by Theorem 3.1.1, and factorizations which correspond to our prediction that n_2 be odd. Further checking verifies that for the representations in the table such that n_2 is odd the factorization corresponds exactly to the prediction of (2.2.3). Therefore, in small cases $\mathcal{B}_2$ exhibits the exotic factorizations we have predicted.

We now move on to the general cases of $\mathcal{B}_n$ and $\mathcal{C}_n$. As we will end up dealing with $\mathcal{B}_n$ and $\mathcal{C}_n$ simultaneously, we will use a bar notation $\overline{\omega}_k$, $\overline{\alpha}_k$, $\overline{V}_k$, etc. for

$n_2 \backslash n_1$	0	1	2	3	4	5	6	7	8	9	10	11	...
0	-	-	-	-	-	-	-	-	-	-	-	-	...
1	-	X	X	X	X	X	X	X	X	X	X	X	...
2	-	-	X	-	-	X	-	-	X	-	-	X	...
3	X	X	X	X	X	X	X	X	X	X	X	X	...
4	-	-	-	-	X	-	-	-	-	X	-	-	...
5	X	X	X	X	X	X	X	X	X	X	X	X	...
6	-	-	-	-	-	-	X	-	-	-	-	-	...
7	X	X	X	X	X	X	X	X	X	X	X	X	...
8	-	-	X	-	-	X	-	-	X	-	-	X	...
9	X	X	X	X	X	X	X	X	X	X	X	X	...
10	-	-	-	-	-	-	-	-	-	-	X	-	...
11	X	X	X	X	X	X	X	X	X	X	X	X	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 2.12: X = $\pi_{[n_1,n_2]}$ factors in $\mathfrak{R}(\mathcal{B}_2)$

objects related to $\mathcal{C}_n$ to avoid confusion. For purposes of our discussion in this section, whenever not explicitly specified, n will be assumed to be ≥ 3 and fixed.

If we try to mimic the basic approach used in $\mathbf{R}_{\mathcal{G}_2}$, $\mathbf{R}_{\mathcal{F}_4}$, and $\mathbf{R}_{\mathcal{B}_2}$, we encounter a new problem. Namely, for $n \geq 3$, $\mathbf{R}_{\mathcal{B}_n}$ no longer contains equal numbers of short and long simple roots; hence our earlier techniques must be modified. Recalling that the exotic admissible map $\varepsilon_{\mathcal{B}_n}$ maps the root lattice of $\mathcal{B}_n$ into the root lattice of $\mathcal{B}_n^* = \mathcal{C}_n$; for $n \geq 2$ these lattices are distinct so the admissible map is no longer an endomorphism. Nevertheless, proceeding as before, there is a unique way to define the admissible map $\varepsilon_{\mathcal{B}_n} : \widetilde{\mathbb{T}}(\mathcal{B}_n) \to \widetilde{\mathbb{T}}(\mathcal{C}_n)$ subject to the length and angle conditions on simple roots introduced earlier. The action of this map and the corresponding maps $\epsilon_{\mathcal{B}_n}$ and $\xi_{\mathcal{B}_n}$ are given by:

α_i	$\varepsilon_{\mathcal{B}_n}\alpha_i$	Weight ω_i	$\epsilon_{\mathcal{B}_n}\omega_i$	Generator X_i	$\xi_{\mathcal{B}_n}X_i$
α_1	$2\overline{\alpha}_1$	ω_1	$2\overline{\omega}_1$	X_1	$\overline{X}_1^2$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
α_{n-1}	$2\overline{\alpha}_{n-1}$	ω_{n-1}	$2\overline{\omega}_{n-1}$	X_{n-1}	$\overline{X}_{n-1}^2$
α_n	$\overline{\alpha}_n$	ω_n	$\overline{\omega}_n$	X_n	$\overline{X}_n$

By a similar argument, the admissible map $\varepsilon_{\mathcal{C}_n} : \widetilde{\mathbb{T}}(\mathcal{C}_n) \to \widetilde{\mathbb{T}}(\mathcal{B}_n)$ and its corresponding maps $\epsilon_{\mathcal{C}_n}$ and $\xi_{\mathcal{C}_n}$ are given by:

α_i	$\varepsilon_{\mathcal{C}_n}\alpha_i$	Weight $\overline{\omega}_i$	$\epsilon_{\mathcal{C}_n}\overline{\omega}_i$	Generator $\overline{X}_i$	$\xi_{\mathcal{C}_n}\overline{X}_i$
$\overline{\alpha}_1$	α_1	$\overline{\omega}_1$	ω_1	$\overline{X}_1$	X_1
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\overline{\alpha}_{n-1}$	α_{n-1}	$\overline{\omega}_{n-1}$	ω_{n-1}	$\overline{X}_{n-1}$	X_{n-1}
$\overline{\alpha}_n$	$2\alpha_n$	$\overline{\omega}_n$	$2\omega_n$	$\overline{X}_n$	X_n^2

Remark. From these actions, it is clear that (1.1.2) is satisfied as well.

Since the exotic maps for $\mathcal{B}_n$ and $\mathcal{C}_n$ are maps from a Lie algebra to its dual Lie algebra, we expect that the corresponding exotic factorizations should somehow involve both $\mathcal{B}_n$ and $\mathcal{C}_n$. To find out if this is indeed the case, and if so, how the two appear in the exotic factorizations, we now calculate the polynomials of irreps in $\mathfrak{R}(\mathcal{B}_3)$ and check when they factor. This leads to Table 2.13 summarizing factorizations in $\mathfrak{R}(\mathcal{B}_3)$ of low-dimensional irreps.

When n_3 is even, the factorizations correspond to the positions of ordinary factorizations like those encountered in $\mathcal{A}_2$, as well as the extra factorizations observed along the main diagonal in $\mathcal{G}_2$. But, similar to the cases of $\mathcal{G}_2$ and $\mathcal{B}_2$, there are far more factorizations than can be accounted for by ordinary factorizations. Therefore we proceed as before to look at the factorizations when n_3 is odd and, as in $\mathcal{G}_2$, at the dimensions of the factors which appear.

Let us first focus on the case when $n_3 = 1$; here we find that $\pi_{\omega_3}(\mathcal{B}_3) = \pi_{[0,0,1]}(\mathcal{B}_3)$ divides all these cases in Table 2.13 including trivially $\pi_{[0,0,1]}(\mathcal{B}_3)$ itself; Table 2.14 summarizes the dimensions of these cofactors.

Since $\pi_{[0,0,1]}(\mathcal{B}_3)$ is the spinor representation, we see that as in the $\mathcal{B}_2$ case the spinor representation divides all irreps of $\mathcal{B}_3$ whose highest weight has odd final index, but unlike in $\mathcal{B}_2$, the dimensions of the cofactors are not dimensions of irreps of $\mathcal{B}_3$. Indeed the smallest cofactor is of dimension 6 which is smaller than the lowest-dimensional irreducible representation of $\mathcal{B}_3$ of dimension 7. If we look at the cofactors for $n_3 = 3$, the situation is no better; the lowest-dimensional cofactor there is that of $\pi_{[0,0,3]}(\mathcal{B}_3)$ which is another 14-dimensional cofactor; and again this is not a dimension of an irreducible representation of $\mathcal{B}_3$.

$n_2\backslash n_1$	0	1	2	3	4	5
0	-	-	-	-	-	-
1	-	-	-	-	-	-
2	-	-	-	-	-	-
3	-	-	-	-	-	-
4	-	-	-	-	-	-
5	-	-	-	-	-	-

$n_3 = 0$

$n_2\backslash n_1$	0	1	2	3	4	5
0	-	X	X	X	X	X
1	X	X	X	X	X	X
2	X	X	X	X	X	X
3	X	X	X	X	X	X
4	X	X	X	X	X	X
5	X	X	X	X	X	X

$n_3 = 1$

$n_2\backslash n_1$	0	1	2	3	4	5
0	-	-	-	-	-	-
1	-	-	-	-	-	-
2	-	-	X	-	-	X
3	-	-	-	-	-	-
4	-	-	-	-	-	-
5	-	-	X	-	-	X

$n_3 = 2$

$n_2\backslash n_1$	0	1	2	3	4	5
0	X	X	X	X	X	X
1	X	X	X	X	X	X
2	X	X	X	X	X	X
3	X	X	X	X	X	X
4	X	X	X	X	X	X
5	X	X	X	X	X	X

$n_3 = 3$

Table 2.13: X = $\pi_{[n_1,n_2,n_3]}(\mathcal{B}_3)$ factors in $\mathfrak{R}(\mathcal{B}_3)$

$n_2 \backslash n_1$	0	1	2	3	4	5	$\ldots$
0	1	6	21	56	126	252	$\ldots$
1	14	64	189	448	924	1728	$\ldots$
2	90	350	924	2016	3900	6930	$\ldots$
3	385	1344	3276	6720	12375	21120	$\ldots$
4	1274	4116	9450	18480	32725	54054	$\ldots$
5	3528	10752	23562	44352	76076	122304	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 2.14: Dimension of cofactor $\frac{\pi_{[n_1,n_2,1]}(\mathcal{B}_3)}{\pi_{[0,0,1]}(\mathcal{B}_3)}$ in $\mathfrak{R}(\mathcal{B}_3)$

Nevertheless, these low-dimensional cofactors are not completely unfamiliar. In particular, recalling that the dimensions of the fundamental representations of $\mathcal{C}_3$ are 6, 14, and 14 which are exactly dimensions of the smallest of these cofactors. Further investigation of $\mathcal{C}_3$ shows that the other cofactor dimensions already calculated are also dimensions of irreps of $\mathcal{C}_3$.

To connect the cofactors to irreps of $\mathcal{C}_3$, we begin by calculating the images of the low-dimensional cofactors of $\pi_{[0,0,1]}(\mathcal{B}_3)$ as elements of $\mathfrak{R}(\mathcal{B}_3)$:

$$cof(\pi_{[1,0,1]}(\mathcal{B}_3)) = V_1 - 1$$

$$cof(\pi_{[0,1,1]}(\mathcal{B}_3)) = V_2 - V_1$$

$$cof(\pi_{[0,0,3]}(\mathcal{B}_3)) = V_3^2 - 2V_2 - V_1 - 1$$

As before, we look at the weights of these cofactors:

Weights of $cof(\pi_{[1,0,1]}(\mathcal{B}_3))$					
$[1,0,0]$	$[-1,1,0]$	$[0,-1,2]$	$[0,1,-2]$	$[1,-1,0]$	$[-1,0,0]$

Weights of $cof(\pi_{[0,1,1]}(\mathcal{B}_3))$					
$[0,1,0]$	$[1,-1,2]$	$[-1,0,2]$	$[1,1,-2]$	$[-1,2,-2]$	$[2,-1,0]$
$[-2,1,0]$	$[1,-2,2]$	$[-1,-1,2]$	$[1,0,-2]$	$[-1,1,-2]$	$[0,-1,0]$
$2\times[0,0,0]$					

Weights of $cof(\pi_{[0,0,3]}(\mathcal{B}_3))$					
$[0,0,2]$	$[0,2,-2]$	$[1,0,0]$	$[-1,1,0]$	$[2,-2,2]$	$[0,-1,2]$
$[2,0,-2]$	$[-2,0,2]$	$[0,1,-2]$	$[-2,2,-2]$	$[1,-1,0]$	$[-1,0,0]$
$[0,-2,2]$	$[0,0,-2]$				

Table 2.15: Weights of low-dimensional cofactors of $\pi_{[0,0,1]}(\mathcal{B}_3)$ in $\mathfrak{R}(\mathcal{B}_3)$

On the other hand, Table 2.16 gives the sets of weights of fundamental $\mathcal{C}_3$ representations.

Comparing these two tables, we notice that the weights of $cof(\pi_{[1,0,1]}(\mathcal{B}_3))$, $cof(\pi_{[0,1,1]}(\mathcal{B}_3))$ and $cof(\pi_{[0,0,3]}(\mathcal{B}_3))$ are obtained from the weights of $\pi_{[1,0,0]}(\mathcal{C}_3)$, $\pi_{[0,1,0]}(\mathcal{C}_3)$ and $\pi_{[0,0,1]}(\mathcal{C}_3)$ respectively by the exotic map $\epsilon_{\mathcal{C}_3}$ constructed above.

Thus, it appears that our exotic factorizations in $\mathfrak{R}(\mathcal{B}_3)$ involve both the spinor representation $\pi_{[0,0,1]}(\mathcal{B}_3)$ and cofactors whose weights are determined by the weights

Weights of $\pi_{[1,0,0]}(\mathcal{C}_3)$					
$[1,0,0]$	$[-1,1,0]$	$[0,-1,1]$	$[0,1,-1]$	$[1,-1,0]$	$[-1,0,0]$

Weights of $\pi_{[0,1,0]}(\mathcal{C}_3)$					
$[0,1,0]$	$[1,-1,1]$	$[-1,0,1]$	$[1,1,-1]$	$[-1,2,-1]$	$[2,-1,0]$
$[-2,1,0]$	$[1,-2,1]$	$[-1,-1,1]$	$[1,0,-1]$	$[-1,1,-1]$	$[0,-1,0]$
$2 \times [0,0,0]$					

Weights of $\pi_{[0,0,1]}(\mathcal{C}_3)$					
$[0,0,1]$	$[0,2,-1]$	$[1,0,0]$	$[-1,1,0]$	$[2,-2,1]$	$[0,-1,1]$
$[2,0,-1]$	$[-2,0,1]$	$[0,1,-1]$	$[-2,2,-1]$	$[1,-1,0]$	$[-1,0,0]$
$[0,-2,1]$	$[0,0,-1]$				

Table 2.16: Weights of fundamental representations of $\mathcal{C}_3$

of irreps of $\mathcal{C}_3$.

In each of the previous groups, our results were easily expressible once we shifted our indexing on our representations. When we do so, the factorizations of $\pi_{[1,0,1]}(\mathcal{B}_3)$, $\pi_{[0,1,1]}(\mathcal{B}_3)$ and $\pi_{[0,0,3]}(\mathcal{B}_3)$ become:

$$\widehat{\pi}_{[2,1,2]}(\mathcal{B}_3) = \widehat{\pi}_{[1,1,2]}(\mathcal{B}_3) \cdot \Xi_{\mathcal{C}_3}\widehat{\pi}_{[2,1,1]}(\mathcal{C}_3)$$

$$\widehat{\pi}_{[1,2,2]}(\mathcal{B}_3) = \widehat{\pi}_{[1,1,2]}(\mathcal{B}_3) \cdot \Xi_{\mathcal{C}_3}\widehat{\pi}_{[1,2,1]}(\mathcal{C}_3)$$

$$\widehat{\pi}_{[1,1,4]}(\mathcal{B}_3) = \widehat{\pi}_{[1,1,2]}(\mathcal{B}_3) \cdot \Xi_{\mathcal{C}_3}\widehat{\pi}_{[1,1,2]}(\mathcal{C}_3)$$

Comparing to the exotic factorizations (2.2.1), (2.2.2), and (2.2.3) already observed in $\mathfrak{R}(\mathcal{G}_2)$, $\mathfrak{R}(\mathcal{F}_4)$, and $\mathfrak{R}(\mathcal{B}_2)$ respectively, the exotic factorizations predicted in $\mathfrak{R}(\mathcal{B}_3)$ take the following general form:

$$\widehat{\pi}_{\epsilon[I]}(\mathcal{B}_3) = \widehat{\pi}_{\epsilon[\rho]}(\mathcal{B}_3) \cdot \Xi\widehat{\pi}_{[I]}(\mathcal{C}_3) \tag{2.2.4}$$

Remark. In the above context it is clear that $\epsilon = \epsilon_{\mathcal{C}_3}$ and $[I]$ and $[\rho]$ msut be weights of $\mathcal{C}_3$ since $\epsilon[I]$ and $\epsilon[\rho]$ are weights of $\mathcal{B}_3$. Similarly, $\Xi = \Xi_{\mathcal{C}_3}$ since it is acting on an irrep of $\mathcal{C}_3$. In general, such context clues will often allow us to drop the references to the groups involved.

Based on how we used the admissible maps to come up with (2.2.4), it is reasonable to assume that reversing the roles of $\mathcal{B}_3$ and $\mathcal{C}_3$ give an similar exotic factorization pattern in $\mathfrak{R}(\mathcal{C}_3)$ as well, leading to the following guess:

$$\widehat{\pi}_{\epsilon[I]}(\mathcal{C}_3) = \widehat{\pi}_{\epsilon[\rho]}(\mathcal{C}_3) \cdot \Xi\widehat{\pi}_{[I]}(\mathcal{B}_3) \tag{2.2.5}$$

Remark. As with (2.2.4), context indicates which maps ϵ and Ξ, and which weights $[I]$ and $[\rho]$ are being used.

Applying (2.2.5) to the case of $\mathcal{C}_3$, we see that the 64-dimensional irrep $\pi_{[1,1,0]}(\mathcal{C}_3)$ should be a common factor and the smallest nontrivial exotic factorization should occur in the 448-dimensional representation $\pi_{[3,1,0]}(\mathcal{C}_3)$. This factorization is easily verified to exist with the weights of the cofactor exactly $\epsilon_{\mathcal{B}_3}$ applied to the weights of $\pi_{[1,0,0]}(\mathcal{B}_3)$.

Thus, as in $\mathfrak{R}(\mathcal{F}_4)$ our predictions of exotic factorizations hold for small examples in $\mathfrak{R}(\mathcal{B}_3)$ and $\mathfrak{R}(\mathcal{C}_3)$; although this does not prove they hold in general, it at least makes them plausible. In fact, seeing as how we have already constructed the admissible maps $\varepsilon_{\mathcal{B}_n}$ and $\varepsilon_{\mathcal{C}_n}$, there is no reason to believe we cannot replace 3 by n in (2.2.4) and (2.2.5) to obtain a more general result. Indeed, replacing 3 by 2 in (2.2.4) exactly gives (2.2.3).

Chapter 3

General Factorization Results

We now prove our observations from the previous sections on the ordinary and exotic factorizations and show how the appear in general. We also explore one other series of factorizations, the ^{LS}-factorizations, which we have actually observed during our work in $\mathfrak{R}(\mathcal{G}_2)$, $\mathfrak{R}(\mathcal{B}_2)$, and $\mathfrak{R}(\mathcal{B}_3)$ but did not discuss in those sections (they correspond to the factorizations observed along the main diagonals of Tables 2.8, 2.12 and 2.13).

Although these factorizations occur in $\mathfrak{R}(G)$, they also occur on the level of characters and indeed our proofs of the existence of the ordinary and exotic factorizations will be done via characters. Further work in $\mathfrak{C}(G)$ will then be used to derive the existence of three more classes of factorizations in $\mathfrak{R}(G)$ including the aforementioned ^{LS}-factorizations as well as two other series whose existence cannot be deduced simply by looking at the factorizations calculated thusfar.

3.1 Factorizations in $\mathfrak{R}(G)$ Arising from Admissible Maps

In §2.1 and §2.2 we saw many examples of factorizations in $\mathfrak{R}(G)$ which were related to the maps of Theorem 1.1.2. We call these factorizations 'ordinary' and 'exotic' according to whether they are related to ordinary Adams' maps or the exotic maps introduced in §1.1.2. We now proceed to show the existences of ordinary factorizations and exotic factorizations in general.

3.1.1 Ordinary Factorizations in $\mathfrak{R}(G)$

Recall that the ordinary factorizations observed in $\mathfrak{R}(\mathcal{A}_2)$ in §2.1.2 took the form:

$$\widehat{\pi}_{m[k_1,k_2]} = \widehat{\pi}_{m[\rho]} \cdot \Psi^m \widehat{\pi}_{[k_1,k_2]}$$

We recall from §1.1.2 that the Ψ^m are endomorphisms of $\mathfrak{R}(G)$ induced by the action of the ordinary admissible maps.

Taking this same basic setup we now let G be arbitrary and let $[I]$ be a weight of G. Then one has:

Theorem 3.1.1. *For each ordinary Adams' operation ψ^m, there is a corresponding series of ordinary factorizations in $\mathfrak{R}(G)$ of the form:*

$$\widehat{\pi}_{m[I]}(G) = \widehat{\pi}_{m[\rho]}(G) \cdot \Psi^m \widehat{\pi}_{[I]}(G) \tag{3.1.1}$$

We will require two lemmas:

Lemma 3.1.1. *In $\mathfrak{E}(G)$ one has:*

$$\psi^m \widehat{\chi}_{[I]}(G) = \frac{E_{m[I]}(G)}{E_{m[\rho]}(G)}$$

Proof. Applying ψ^m to both sides of the Weyl Character Formula expression for $\widehat{\chi_{[I]}}(G)$ and noting that $E_{[I]}(G) \in \mathfrak{E}(G)$, one has:

$$\begin{aligned}\psi^m \widehat{\chi}_{[I]}(G) &= \psi^m \frac{E_{[I]}(G)}{E_{[\rho]}(G)} \\ &= \frac{\psi^m E_{[I]}(G)}{\psi^m E_{[\rho]}(G)}\end{aligned}$$

Now since ψ^m is an admissible map it pseudo-commutes with the action of $W(G)$ on any weight $[I]$ so that one has:

$$\begin{aligned}\psi^m E_{[I]}(G) &= \psi^m \sum_{w \in W(G)} (-1)^w \exp(w \circ [I]) \\ &= \sum_{w \in W(G)} (-1)^w \exp(m(w \circ [I])) \\ &= \sum_{w' \in W(G)} (-1)^{w'} \exp(w' \circ (m[I])) \\ &= E_{m[I]}(G)\end{aligned}$$

This proves the lemma. □

The second lemma was mentioned without proof in §1.1.2.

Lemma 3.1.2. *The Adams operations ψ^m lift to homomorphisms $\Psi^m : \mathfrak{R}(G) \to \mathfrak{R}(G)$ such that if $\chi \in \mathfrak{E}(G)$ is the character of some $\pi \in \mathfrak{R}(G)$ then $\Psi^k \pi$ is the element of $\mathfrak{R}(G)$ whose character is $\psi^m \chi$.*

Proof. If χ is the character of $\pi \in \mathfrak{R}(G)$, then it is $W(G)$-symmetric, and clearly $\psi^k\chi$ is also $W(G)$-symmetric, hence is also a character. We claim this character is actually the character of an element of $\mathfrak{R}(G)$ (i.e. is an integer combination of irreps as opposed to a rational combination for example). Note that the k^{th} exterior power $\Lambda^k\pi$ is a representation of G and hence an element of $\mathfrak{R}(G)$ with corresponding character $\lambda^k\chi$. On the other hand, $\lambda^k\chi$ is the k^{th} elementary symmetric polynomial in the monomial terms of χ and $\psi^n\chi$ is the n^{th} power sum of these monomial terms. By Newton's identities the power sums are $\mathbb{Z}$-polynomials in the elementary symmetric polynomials; hence $\Psi^n\pi$ is a $\mathbb{Z}$-polynomial in the representations $\Lambda^k\mu$ so is itself an element of $\mathfrak{R}(G)$.

As $\psi^m : \mathfrak{C}(G) \to \mathfrak{C}(G)$ is clearly a homomorphism and direct sums and tensor products of virtual representations in $\mathfrak{R}(G)$ correspond to addition and multiplication of characters in $\mathfrak{C}(G)$, it follows that $\Psi^m : \mathfrak{R}(G) \to \mathfrak{R}(G)$ is also a homomorphism. □

We are now ready to prove Theorem 3.1.1 about ordinary factorizations in $\mathfrak{R}(G)$.

Proof. First translate the statement to the corresponding statement on characters:

$$\widehat{\chi}_{m[I]}(G) = \widehat{\chi}_{m[\rho]}(G) \cdot \psi^m \widehat{\chi}_{[I]}(G)$$

The Weyl Character Formula on the RHS gives:

$$\begin{aligned}\widehat{\chi}_{m[I]}(G) &= \frac{E_{m[I]}(G)}{E_{[\rho]}(G)} \\ &= \frac{E_{m[\rho]}(G)}{E_{[\rho]}(G)} \cdot \frac{E_{m[I]}(G)}{E_{m[\rho]}(G)} \end{aligned} \tag{3.1.2}$$

By the Weyl Character Formula, the first factor on the RHS of (3.1.2) is $\widehat{\chi}_{m[\rho]}(G)$. By Lemma 3.1.1 the second factor of (3.1.2) is $\psi^m \widehat{\chi}_{[I]}(G)$. Thus by the Weyl Character Formula and Lemma 3.1.2, (ref21) is equal to the LHS of (3.1.1) which proves the theorem. $\square$

Remark. One immediate consequence of Theorem 3.1.1 is that irreducible representations of the form $\pi_{m[\rho]}(G)$ divide all irreps whose highest weight lies in a certain sublattice of the weight lattice and the resulting cofactors are virtual representations obtained by applying the Adams' operations to other irreps.

Note that Theorem 3.1.1 applied to $\mathcal{A}_1$ explains some, but not all, of the observed factorizations in $\mathfrak{R}(\mathcal{A}_1)$. Furthermore, Theorem 3.1.1 predicts multiple factorizations of some irreps. Both these issues will be dealt with in §3.2.1.

3.1.2 Exotic Factorizations in $\mathfrak{R}(G)$

We now show that in general when G is non-simply laced, there is a series of factorizations related to the exceptional admissible maps of Theorem 1.1.2. In particular, we show that the patterns of exotic factorizations observed in $\mathfrak{R}(\mathcal{B}_2)$ and $\mathfrak{R}(\mathcal{G}_2)$ continue and that the predicted factorizations in $\mathfrak{R}(\mathcal{F}_4)$, $\mathfrak{R}(\mathcal{B}_n)$, and $\mathfrak{R}(\mathcal{C}_n)$ also appear in general. To keep our expressions from becoming too unwieldly, we will drop the reference to the group from the maps induced by ε_G (so ϵ is taken to mean ϵ_G and so forth) and use the shorthand notation ε^* to mean ε_{G^*} and similarly for the notations ϵ^*, ξ^* and Ξ^*.

Theorem 3.1.2. *Let G be a nonsimply-laced Lie group and ϵ, ξ, and Ξ the maps (whose actions were defined in §2.2) corresponding to the exotic admissible map $\varepsilon_G : \widetilde{\mathbb{T}}(G) \to \widetilde{\mathbb{T}}(G^*)$. In addition to the ordinary factorizations $\mathfrak{R}(G)$, there is an additional series of exotic factorizations among the irreps in $\mathfrak{R}(G)$ of the following form:*

$$\widehat{\pi}_{\epsilon^*[I]}(G) = \widehat{\pi}_{\epsilon^*[\rho]}(G) \cdot \Xi^* \widehat{\pi}_{[I]}(G^*) \tag{3.1.3}$$

The proof is similar to that of Theorem 3.1.1 and requires an analogous form of Lemma 3.1.1.

Lemma 3.1.3. *When G is nonsimply-laced, in $\mathfrak{E}(G)$ one has:*

$$\xi^* \widehat{\chi}_{[I]}(G^*) = \frac{E_{\epsilon^*[I]}(G)}{E_{\epsilon^*[\rho]}(G)}$$

Proof. Applying ξ^* to both sides of the Weyl Character Formula expression for $\widehat{\chi}_{[I]}(G^*)$, one has:

$$\begin{aligned} \xi^* \widehat{\chi}_{[I]}(G^*) &= \xi^* \frac{E_{[I]}(G^*)}{E_{[\rho]}(G^*)} \\ &= \frac{\xi^* E_{[I]}(G^*)}{\xi^* E_{[\rho]}(G^*)} \end{aligned} \tag{3.1.4}$$

Now since ε^* is admissible, for any weight $[I]$ of G^*, one has:

$$\begin{aligned}
\xi^* E_{[I]}(G^*) &= \xi^* \sum_{w \in W(G^*)} (-1)^w \exp(w \circ [I]) \\
&= \sum_{w \in W(G^*)} (-1)^w \exp(\epsilon^*(w \circ [I])) \qquad (3.1.5) \\
&= \sum_{w' \in W(G)} (-1)^{w'} \exp(w' \circ (\epsilon^*[I])) \\
&= E_{\epsilon^*[I]}(G) \qquad (3.1.6)
\end{aligned}$$

Remark. (3.1.5) follows from the fact that for the fundamental weights ω_k^* of G^* one clearly has $\xi^*(\exp(\omega_k^*)) = \exp(\varepsilon^* \omega_k^*)$.

Equality (3.1.6) allows one to rewrite (3.1.4) as $\frac{E_{\epsilon^*[I]}(G)}{E_{\epsilon^*[\rho]}(G)}$ which completes the proof. $\square$

We are now able to prove Theorem 3.1.2

Proof. To prove the theorem, as with Theorem 3.1.1 we first translate the statement to the corresponding statement on characters:

$$\widehat{\chi}_{\epsilon^*[I]}(G) = \widehat{\chi}_{\epsilon^*[\rho]}(G) \cdot \xi^* \widehat{\chi}_{[I]}(G^*)$$

By the Weyl Character Formula we have:

$$\begin{aligned}
\widehat{\chi}_{\epsilon^*[I]}(G) &= \frac{E_{\epsilon^*[I]}(G)}{E_{[\rho]}(G)} \\
&= \frac{E_{\epsilon^*[\rho]}(G)}{E_{[\rho]}(G)} \cdot \frac{E_{\epsilon^*[I]}(G)}{E_{\epsilon^*[\rho]}(G)}
\end{aligned}$$

The first factor is clearly $\widehat{\chi}_{\epsilon^*[\rho]}$. By Lemma 3.1.3 the second factor is $\xi^*\widehat{\chi}_{[I]}(G^*)$. It remains to show that $\xi^*\widehat{\chi}_{[I]}(G^*)$ is the character of some virtual representation in $\mathfrak{R}(G)$. Since $\widehat{\chi}_{[I]}(G^*)$ is the character of a representation of G^*, its set of weights is $W(G^*)$-symmetric. Now since ϵ^* is a lattice homomorphism from the weight lattice of G^* to the weight lattice of G, it follows that the set of weights of $\xi^*\widehat{\chi}_{[I]}(G^*)$ is $W(G)$-symmetric and the coefficients of these weights are integers (since the multiplicities of weights of $\widehat{\chi}_{[I]}(G^*)$ are integers and the action of ϵ^* clearly does not affect multiplicities). Thus $\xi^*\widehat{\chi}_{[I]}(G^*) \in \mathfrak{E}(G)^{W(G)}$, hence is the character of some virtual representation in $\mathfrak{R}(G)$ as claimed. $\square$

Remark. From the above, we know that $\Xi^*\widehat{\pi}_{[I]}(G^*)$ is some virtual representation of G, but we do not know anything about its image in $\mathfrak{R}(G)$. Using what is known about the action of ξ on $\mathfrak{E}(G)$ one can work out $\xi^*\widehat{\chi}_{[I]}(G^*)$; applying Algorithm 1.1.1 gives $\Xi^*\widehat{\pi}_{[I]}(G^*)$ as an element of $\mathfrak{R}(G)$.

3.2 Factorizations in $\mathfrak{R}(G)$ Arising from Factorization Results in $\mathfrak{E}(G)$

We now discuss a number of other factorizations which are not connected to admissible maps. Instead, as we will see, these factorizations arise naturally from considering factorizations from the standpoint of the geometry of the root system $\mathbf{R_G}$ to obtain factorizations in $\mathfrak{E}(G)$; using unique factorization in $\mathfrak{R}(G)$ we then use these factorizations to deduce the existence of new factorizations in $\mathfrak{R}(G)$.

Because $\mathfrak{E}(G)$ is a Laurent polynomial ring we must be somewhat careful to specify what exactly is meant by a factorization in $\mathfrak{E}(G)$. For our purposes, $p \in \mathfrak{E}(G)$ factors iff when written as $p = \frac{q(X_1,\ldots,X_n)}{X_1^{k_1}\ldots X_n^{k_n}}$ in reduced form, the polynomial q factors in $\mathbb{Z}[X_1, \ldots, X_n]$. Since each X_i is a unit in $\mathfrak{E}(G)$, we may break up the denominator up among the different factors of the numerator however we like without affecting the factorization of p. For example, $X_1 - X_1^{-1}$ factors since it equals $\frac{X_1^2-1}{X_1}$ and $X_1^2 - 1$ factors in $\mathbb{Z}[X_1]$.

As we will see, in our factorizations it is actually more convenient to allow half-integer exponents when breaking up the denominator of an element among factors; we already used such a factorization in our statement of the Denominator Formula (Theorem 1.1.5). Although formally elements such as $X_i^{\frac{1}{2}}$ do not exist in $\mathfrak{E}(G)$, their appearance in our factorizations of characters will not affect our final results. Thus for example we have the following two possible factorizations of $X_1 - X_1^{-1}$:

$$X_1 - X_1^{-1} = \frac{X_1 + 1}{X_1^{\frac{1}{2}}} \cdot \frac{X_1 - 1}{X_1^{\frac{1}{2}}} \tag{3.2.1}$$

$$X_1 - X_1^{-1} = \frac{X_1 + 1}{X_1^{\frac{1}{2}}} \cdot \frac{X_1^{\frac{1}{2}} + 1}{X_1^{\frac{1}{2}}} \cdot (X_1^{\frac{1}{2}} - 1) \tag{3.2.2}$$

Even though formally neither factor in (3.2.1) is an element of $\mathfrak{E}(G)$, we allow it as a valid factorization because both the numerators involve only integer exponents. On the other hand, we do not consider (3.2.2) as a valid factorization because the last two factors involve half-integer exponents.

3.2.1 Γ-factorizations

Thusfar we have been primarily concerned with factorizations in $\mathfrak{R}(G)$. The factors guaranteed by Theorems 3.1.1 and 3.1.2 give rise to factors of the corresponding irreducible characters in $\mathfrak{C}(G)$; such factors are necessarily $W(G)$-symmetric (since they are class functions on G). However, when we examine factorization in $\mathfrak{C}(G)$, we find that there are irreducible characters which factor in $\mathfrak{C}(G)$ even though the corresponding irreps do not factor in $\mathfrak{R}(G)$. For a nontrivial example of this phenomenon, consider the factorization table of irreducible characters in $\mathfrak{C}(\mathcal{A}_2)$ given in Table 3.1.

The difference between Tables 2.5 and 3.1 is along the main diagonal; in Table 3.1 we see that for $n \geq 0$ all $\chi_{[n,n]}(\mathcal{A}_2)$ factor in $\mathfrak{C}(\mathcal{A}_2)$ whereas by Table 2.5 only some $\pi_{[n,n]}(\mathcal{A}_2)$ factor in $\mathfrak{R}(\mathcal{A}_2)$. The reason for this difference becomes apparent when we consider these characters from the viewpoint of the Weyl Character Formula. In particular, one has $\widehat{\chi}_{[n,n]}(\mathcal{A}_2) = \frac{E_{[n,n]}(\mathcal{A}_2)}{E_{[0,0]}(\mathcal{A}_2)}$.

The numerator of this expression factors in $\mathfrak{C}(\mathcal{A}_2)$:

$$\begin{aligned} E_{[n,n]}(\mathcal{A}_2) &= X_1^n X_2^n + X_1^n X_2^{-2n} + X_1^{-2n} X_2^n \\ &\quad - X_1^{-n} X_2^{-n} - X_1^{-n} X_2^{2n} - X_1^{2n} X_2^{-n} \\ &= \frac{(X_1^{2n} - X_2^n)(X_2^{2n} - X_1^n)(X_1^n X_2^n - 1)}{X_1^{2n} X_2^{2n}} \end{aligned}$$

If we want to include the denominator among these factors, the most natural

$n_2 \backslash n_1$	0	1	2	3	4	5	6	7	8	9	10	11	$\dots$
0	-	-	-	-	-	-	-	-	-	-	-	-	$\dots$
1	-	X	-	X	-	X	-	X	-	X	-	X	$\dots$
2	-	-	X	-	-	X	-	-	X	-	-	X	$\dots$
3	-	X	-	X	-	X	-	X	-	X	-	X	$\dots$
4	-	-	-	-	X	-	-	-	-	X	-	-	$\dots$
5	-	X	X	X	-	X	-	X	X	X	-	X	$\dots$
6	-	-	-	-	-	-	X	-	-	-	-	-	$\dots$
7	-	X	-	X	-	X	-	X	-	X	-	X	$\dots$
8	-	-	X	-	-	X	-	-	X	-	-	X	$\dots$
9	-	X	-	X	X	X	-	X	-	X	-	X	$\dots$
10	-	-	-	-	-	-	-	-	-	-	X	-	$\dots$
11	-	X	X	X	-	X	-	X	X	X	-	X	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 3.1: X = Irreducible Character $\chi_{[n_1,n_2]}$ factors in $\mathfrak{E}(\mathcal{A}_2)$

choice is to split the denominator so as to symmetrize the factors:

$$E_{[n,n]}(A_2) = (X_1^n X_2^{-\frac{n}{2}} - X_1^{-n} X_2^{\frac{n}{2}}) \cdot (X_1^{-\frac{n}{2}} X_2^n - X_1^{\frac{n}{2}} X_2^{-n}) \cdot (X_1^{\frac{n}{2}} X_2^{\frac{n}{2}} - X_1^{-\frac{n}{2}} X_2^{-\frac{n}{2}})$$

The advantage of the above symmetrization is that the resulting factors form an $W(\mathcal{A}_2)$-alternating set in $\mathfrak{E}(\mathcal{A}_2)$; however the factors are not guaranteed to be characters of elements of $\mathfrak{R}(\mathcal{A}_2)$.

Nevertheless, this factorization in $\mathfrak{E}(G)$ is exactly what we need; we find that there are as many factors of $E_{[n,n]}(\mathcal{A}_2)$ as elements of $\mathbf{R}^+_{\mathcal{A}_2}$ and these factors are easily described. In fact an analogous statement holds for arbitrary Lie groups:

Theorem 3.2.1. *In $\mathfrak{E}(G)$, $E_{m[\rho]}(G)$ factors as:*

$$E_{m[\rho]}(G) = \prod_{r \in \mathbf{R}^+_{\mathbf{G}}} (X^{m \cdot v(r)} - X^{-m \cdot v(r)})$$

Proof. This is a consequence of the Weyl Denominator Formula:

$$\begin{aligned} E_{m[\rho]}(G) &= \psi^m E_{[\rho]}(G) \\ &= \psi^m \prod_{r \in \mathbf{R}^+_{\mathbf{G}}} (X^{v(r)} - X^{-v(r)}) \\ &= \prod_{r \in \mathbf{R}^+_{\mathbf{G}}} \psi^m (X^{v(r)} - X^{-v(r)}) \\ &= \prod_{r \in \mathbf{R}^+_{\mathbf{G}}} (X^{m \cdot v(r)} - X^{-m \cdot v(r)}) \end{aligned}$$

□

Corollary 3.2.1. *$\widehat{\chi}_{m[\rho]}(G)$ factors in $\mathfrak{E}(G)$ for $m \geq 2$ with the sole exception $\widehat{\chi}_{[2]}(\mathcal{A}_1) = \widehat{\chi}_{2[\rho]}(\mathcal{A}_1)$. The individual factors are not characters of elements of $\mathfrak{R}(G)$ unless $G = \mathcal{A}_1$.*

Proof. The Character Formula and Theorem 3.2.1 give:

$$\begin{aligned} \widehat{\chi}_{m[\rho]}(G) &= \frac{E_{m[\rho]}(G)}{E_{[\rho]}(G)} \\ &= \frac{\prod_{r \in \mathbf{R}^+_{\mathbf{G}}} (X^{m \cdot v(r)} - X^{-m \cdot v(r)})}{\prod_{r \in \mathbf{R}^+_{\mathbf{G}}} (X^{v(r)} - X^{-v(r)})} \end{aligned}$$

For each $r \in \mathbf{R}^+_{\mathbf{G}}$, $(X^{v(r)} - X^{-v(r)})$ divides $(X^{m \cdot v(r)} - X^{-m \cdot v(r)})$ in $\mathfrak{E}(G)$. The set of cofactors $\frac{X^{m \cdot v(r)} - X^{-m \cdot v(r)}}{X^{v(r)} - X^{-v(r)}}$ are clearly a $W(G)$-symmetric set and there are $|\mathbf{R}^+_{\mathbf{G}}|$ such cofactors, hence $\widehat{\chi}_{m[\rho]}(G)$ has at least $|\mathbf{R}^+_{\mathbf{G}}|$ factors. In the case $G = \mathcal{A}_1$ where $|\mathbf{R}^+_{\mathcal{A}_1}| = \mathbf{1}$, this approach fails.

On the other hand, our work in §2.1.1 shows that irreducible characters of $\mathcal{A}_1$ factor into $W(\mathcal{A}_1)$-symmetric factors, hence each such factor is itself the character of some element of $\mathfrak{R}(\mathcal{A}_1)$. $\square$

With these results at hand, we return to Theorem 3.1.1 which in some cases predicts multiple factorization of a given irrep. As $\mathfrak{R}(G)$ is a UFD, we conclude that the ordinary factorizations found in these cases split further. For example, by Theorem 3.1.1 we have the following two factorizations of $\widehat{\pi}_{[6,6]}(\mathcal{A}_2)$:

$$\begin{aligned}\widehat{\pi}_{[6,6]}(\mathcal{A}_2) &= \widehat{\pi}_{[2,2]}(\mathcal{A}_2) \cdot \psi^2 \widehat{\pi}_{[3,3]}(\mathcal{A}_2) \\ &= \widehat{\pi}_{[3,3]}(\mathcal{A}_2) \cdot \psi^3 \widehat{\pi}_{[2,2]}(\mathcal{A}_2)\end{aligned}$$

It is easy to check that neither of $\widehat{\pi}_{[2,2]}(\mathcal{A}_2)$ and $\widehat{\pi}_{[3,3]}(\mathcal{A}_2)$ divides the other, so we must be able to decompose both these factorizations further. In light of Theorem 3.2.1, in $\mathfrak{E}(\mathcal{A}_2)$ we have:

$$\begin{aligned}E_{[n,n]}(\mathcal{A}_2) = \prod_{d|n} \Big(&\Phi_d(X_1 X_2^{-\frac{1}{2}}, X_1^{-1} X_2^{\frac{1}{2}}) \cdot \Phi_d(X_1^{-\frac{1}{2}} X_2, X_1^{\frac{1}{2}} X_2^{-1}) \\ &\cdot \Phi_d(X_1^{\frac{1}{2}} X_2^{\frac{1}{2}}, X_1^{-\frac{1}{2}} X_2^{-\frac{1}{2}}) \Big)\end{aligned}$$

In particular, we therefore have:

$$\widehat{\chi}_{[2,2]}(\mathcal{A}_2) = \Phi_2(X_1X_2^{-\frac{1}{2}}, X_1^{-1}X_2^{\frac{1}{2}}) \cdot \Phi_2(X_1^{-\frac{1}{2}}X_2, X_1^{\frac{1}{2}}X_2^{-1}) \cdot \Phi_2(X_1^{\frac{1}{2}}X_2^{\frac{1}{2}}, X_1^{-\frac{1}{2}}X_2^{-\frac{1}{2}})$$

$$\widehat{\chi}_{[3,3]}(\mathcal{A}_2) = \Phi_3(X_1X_2^{-\frac{1}{2}}, X_1^{-1}X_2^{\frac{1}{2}}) \cdot \Phi_3(X_1^{-\frac{1}{2}}X_2, X_1^{\frac{1}{2}}X_2^{-1}) \cdot \Phi_3(X_1^{\frac{1}{2}}X_2^{\frac{1}{2}}, X_1^{-\frac{1}{2}}X_2^{-\frac{1}{2}})$$

$$\widehat{\chi}_{[6,6]}(\mathcal{A}_2) = \prod_{d=2,3,6} \Phi_d\Big((X_1X_2^{-\frac{1}{2}}, X_1^{-1}X_2^{\frac{1}{2}}) \cdot \Phi_d(X_1^{-\frac{1}{2}}X_2, X_1^{\frac{1}{2}}X_2^{-1}) \cdot \Phi_d(X_1^{\frac{1}{2}}X_2^{\frac{1}{2}}, X_1^{-\frac{1}{2}}X_2^{-\frac{1}{2}})\Big)$$

It is also easy to check that:

$$\psi^3\widehat{\chi}_{[2,2]}(\mathcal{A}_2) = \prod_{d=2,6} \Big(\Phi_d(X_1X_2^{-\frac{1}{2}}, X_1^{-1}X_2^{\frac{1}{2}}) \cdot \Phi_d(X_1^{-\frac{1}{2}}X_2, X_1^{\frac{1}{2}}X_2^{-1}) \cdot \Phi_d(X_1^{\frac{1}{2}}X_2^{\frac{1}{2}}, X_1^{-\frac{1}{2}}X_2^{-\frac{1}{2}})\Big)$$

$$\psi^2\widehat{\chi}_{[3,3]}(\mathcal{A}_2) = \prod_{d=3,6} \Big(\Phi_d(X_1X_2^{-\frac{1}{2}}, X_1^{-1}X_2^{\frac{1}{2}}) \cdot \Phi_d(X_1^{-\frac{1}{2}}X_2, X_1^{\frac{1}{2}}X_2^{-1}) \cdot \Phi_d(X_1^{\frac{1}{2}}X_2^{\frac{1}{2}}, X_1^{-\frac{1}{2}}X_2^{-\frac{1}{2}})\Big)$$

Hence we do indeed have a unique factorization of $\widehat{\chi}_{[6,6]}(\mathcal{A}_2)$ in $\mathfrak{E}(\mathcal{A}_2)$. Upon factoring the $\widehat{\chi}_{[2,2]}(A_2)$ and $\widehat{\chi}_{[3,3]}(A_2)$ factors from $\widehat{\chi}_{[6,6]}(A_2)$, we are left with a new cofactor (whose dimension must be $1 = \frac{\dim(\widehat{\pi}_{[6,6]})}{\dim(\widehat{\pi}_{[2,2]})\cdot\dim(\widehat{\pi}_{[3,3]})}$):

$$\gamma_6(\mathcal{A}_2) := \Phi_6(X_1X_2^{-\frac{1}{2}}, X_1^{-1}X_2^{\frac{1}{2}}) \cdot \Phi_6(X_1^{-\frac{1}{2}}X_2, X_1^{\frac{1}{2}}X_2^{-1}) \cdot \Phi_6(X_1^{\frac{1}{2}}X_2^{\frac{1}{2}}, X_1^{-\frac{1}{2}}X_2^{-\frac{1}{2}})$$

The factors of $\gamma_6(\mathcal{A}_2)$ clearly form a $W(\mathcal{A}_2)$-symmetric set, thus $\gamma_6(\mathcal{A}_2)$ is $W(\mathcal{A}_2)$-symmetric. Furthermore, $\gamma_6(\mathcal{A}_2)$ lies in $\mathfrak{E}(A_2)$ since the degree of Φ_6 is even, so it is the character of some virtual representation $\Gamma_6(\mathcal{A}_2) \in \mathfrak{R}(\mathcal{A}_2)$. By construction, $\Gamma_6(\mathcal{A}_2)$ is a factor of $\widehat{\pi}_{[6,6]}(A_2)$ whose existence is only implied by Theorem 3.1.1 combined with unique factorization. From the character $\gamma_6(\mathcal{A}_2)$, one computes:

$$\Gamma_6(\mathcal{A}_2) = V_1^2 V_2^2 - 3V_1^3 - 3V_2^3 + 10V_1V_2 - 8$$

Specialization at $V_1 = V_2 = 3$ verifies that the dimension of Γ_6 is indeed 1.

By the same basic setup as above, there are many examples of such extra 1-dimensional factors in $\mathfrak{R}(G)$ in general as we now show.

Definition 3.2.1. *With $v(r)$ defined as in Theorem 1.1.5 define $\gamma_d(G) \in \mathfrak{E}(G)$ to be:*

$$\gamma_d(G) := \begin{cases} 1 & d = 1 \\ \prod_{r \in \mathbf{R}_\mathbf{G}^+} \Phi_d(X^{v(r)}, X^{-v(r)}) & d \geq 2 \end{cases}$$

Further define elements $\Gamma_d(G) \in \mathfrak{R}(G)$ by the condition that $\gamma_d(G)$ is the character of $\Gamma_d(G)$.

The set of factors of $\gamma_d(G)$ clearly form a $W(G)$-symmetric set, so in particular $\gamma_d(G)$ is $W(G)$-symmetric which shows that it makes sense to define the $\Gamma_d(G)$ in this way.

Theorem 3.2.2. *(Γ-factorizations in $\mathfrak{R}(G)$)*

The irreducible character $\widehat{\chi}_{m[\rho]}(G)$ factors in $\mathfrak{E}(G)$ as:

$$\widehat{\chi}_{m[\rho]}(G) = \prod_{d|m} \gamma_d(G) \tag{3.2.3}$$

Furthermore, (3.2.3) lifts to a factorization of $\widehat{\pi}_{m[\rho]}(G)$ in $\mathfrak{R}(G)$ of the following form:

$$\widehat{\pi}_{m[\rho]}(G) = \prod_{d|m} \Gamma_d(G)$$

Proof. Breaking down the factors of Theorem 3.2.1 into their cyclotomic factors, one has:

$$E_{m[\rho]}(G) = \prod_{r\in\mathbf{R}_{\mathbf{G}}^{+}} \prod_{d|m} \Phi_d(X^{v(r)}, X^{-v(r)})$$

Dividing both sides by $E_{[\rho]}(G)$ gives:

$$\begin{aligned}
\widehat{\chi}_{m[\rho]}(G) &= \prod_{r\in\mathbf{R}_{\mathbf{G}}^{+}} \prod_{1<d,d|m} \Phi_d(X^{v(r)}, X^{-v(r)}) \\
&= \prod_{1<d,d|m} \gamma_d(G) \\
&= \prod_{d|m} \gamma_d(G)
\end{aligned}$$

For d fixed, the set of factors $\Phi_d(X^{v(r)}, X^{-v(r)})$ clearly forms a $W(G)$-symmetric set; thus $\gamma_d(G)$ is $W(G)$-symmetric. Hence $\gamma_d(G)$ is the character of some $\Gamma_d(G) \in \mathfrak{R}(G)$. $\square$

Remark. In the case of $\mathcal{A}_1$ where there is only one positive root, one has $\gamma_d(\mathcal{A}_1) = \Phi_d(X_1, X_1^{-1})$. Thus the factorizations in Table 2.3 are exactly the Γ-factorizations in $\mathfrak{R}(\mathcal{A}_1)$.

Corollary 3.2.2. *The virtual dimension of* $\Gamma_d(G)$ *is given by:*

$$\dim(\Gamma_d(G)) = \begin{cases} p^{|\mathbf{R}^+_\mathbf{G}|} & d = p^k \textit{ with } p \textit{ prime} \\ \\ 1 & \textit{otherwise} \end{cases}$$

Proof. The virtual dimension of each factor $\Phi_d(X^{v(r)}, X^{-v(r)})$ is obtained by specializing at $\{X_i = 1\}_{i=1}^n$ which clearly equals $\Phi_d(1)$. Since there are $|\mathbf{R}^+_\mathbf{G}|$ such factors of $\Gamma_d(G)$, the result follows. $\square$

In the case that d is prime, one clearly has $\Gamma_d(G) = \widehat{\pi}_{d[\rho]}(G)$. More generally, when $d = p^k$ for p a prime, one has:

$$\Gamma_{p^k}(G) = \Psi^{p^{k-1}} \widehat{\pi}_{d[\rho]}(G)$$

This is easily shown by induction on k combined with Theorem 3.1.1.

Finally, we note that for d not a power of a prime, although $\Gamma_d(G)$ has virtual dimension 1, it is clearly not the trivial representation, hence in such cases $\Gamma_d(G)$ is a nontrivial factor of $\widehat{\pi}_{d[\rho]}(G)$ in $\mathfrak{R}(G)$.

We have now seen how working in $\mathfrak{E}(G)$ allows us to discover even more factorization results in $\mathfrak{R}(G)$.

3.2.2 ^{LS}-factorizations

In this section we will further refine the techniques used in §3.2.1 to obtain further factorization results in $\mathfrak{R}(G)$ when G is nonsimply-laced.

As we saw in Theorem 1.1.5, $E_{m[\rho]}(G)$ factors in $\mathfrak{E}(G)$ for any G and furthermore if we allow half-integer exponents then we can associate positive roots of G with factors of $E_{m[\rho]}(G)$ in a natural way. In the case of the nonsimply-laced groups, factorization in $\mathfrak{E}(G)$ can again be used to deduce factorization results in $\mathfrak{R}(G)$ not already predicted by any of Theorems 3.1.1, 3.1.2, or 3.2.2.

Theorem 3.2.3. *If G is nonsimply-laced, then $\widehat{\pi}_{m[\rho]}(G)$ factors in $\mathfrak{R}(G)$ for all $k \geq 1$.*

Proof. Let G be a fixed nonsimply-laced group. Let $\mathbf{R}_\mathbf{G}^\mathbf{L}$ and $\mathbf{R}_\mathbf{G}^\mathbf{S}$ denote the subsets of $\mathbf{R}_\mathbf{G}^+$ consisting respectively of the long and short positive roots of G. By Theorem 3.2.1, $E_{m[\rho]}(G)$ factors in $\mathfrak{E}(G)$ into factors which are in correspondence with elements of $\mathbf{R}^+$. Note that since elements of $W(G)$ do not change the lengths of roots, the factors associated to elements of $\mathbf{R}_\mathbf{G}^\mathbf{L}$ form a $W(G)$-alternating subset of all the factors and likewise for the set of factors associated to elements of $\mathbf{R}_\mathbf{G}^\mathbf{S}$. Now with v as in Theorem 1.1.5, set:

$$E^L_{m[\rho]}(G) := \prod_{r \in \mathbf{R}_\mathbf{G}^\mathbf{L}} (X^{m \cdot v(r)} - X^{-m \cdot v(r)})$$

$$E^S_{m[\rho]}(G) := \prod_{r \in \mathbf{R}_\mathbf{G}^\mathbf{S}} (X^{m \cdot v(r)} - X^{-m \cdot v(r)})$$

Thus both $E^L_{m[\rho]}(G)$ and $E^S_{m[\rho]}(G)$ are each $W(G)$-alternating. Define further:

$$\widehat{\chi}^L_{m[\rho]}(G) := \frac{E^L_{m[\rho]}(G)}{E^L_{[\rho]}(G)}$$

$$\widehat{\chi}^S_{m[\rho]}(G) := \frac{E^S_{m[\rho]}(G)}{E^S_{[\rho]}(G)}$$

Then $\widehat{\chi}^{L}_{m[\rho]}(G)$ and $\widehat{\chi}^{S}_{m[\rho]}(G)$ are $W(G)$-symmetric and are clearly elements of $\mathfrak{E}(G)$ (since they are products of factors of the form $\Phi_d(X^{v(r)}, X^{-v(r)})$), hence are characters of representations $\widehat{\pi}^{L}_{m}(G)$ and $\widehat{\pi}^{S}_{m}(G)$ in $\mathfrak{R}(G)$. Then by construction, the desired factorization of $\widehat{\pi}_{m[\rho]}(G)$ in $\mathfrak{R}(G)$ is given by:

$$\widehat{\pi}_{m[\rho]}(G) = \widehat{\pi}^{L}_{m}(G) \cdot \widehat{\pi}^{S}_{m}(G) \tag{3.2.4}$$

□

Remark. Recall that Theorem 3.1.1 only guaranteed factorizations of $\widehat{\pi}_{m[\rho]}(G)$ when m is composite. Theorem 3.2.2 guaranteed more factors of $\widehat{\pi}_{m[\rho]}(G)$ beyond those of Theorem 3.1.1 when m is composite. Theorem 3.1.2 further guarantees a factorization of $\widehat{\pi}_{q[\rho]}(G)$ where q is the characteristic of G. Thus, the factorizations of Theorem 3.2.3 give many new factorizations in $\mathfrak{R}(G)$ not already covered.

Applying Theorem 3.2.3 to $\mathcal{G}_2$ explains the factorizations appearing along the main diagonal of Table 2.8 and similarly applying it to $\mathcal{B}_2$ explains the extra main diagonal factorizations of Table 2.12.

For convenience, we call $\widehat{\pi}^{L}_{m}(G)$ the 'long factor' of $\widehat{\pi}_{m[\rho]}(G)$ and similarly $\widehat{\pi}^{S}_{m}(G)$ is the 'short factor'. Because they come from considerations of long and short roots, we thus call the factorizations of (3.2.4) ^{LS}-factorizations.

Since the $\mathfrak{E}(G)$-factorization of the long and short factors involves the long and short roots, it is natural to ask what relationship the exotic maps have to these factors. In fact, the relationship is quite simple:

Theorem 3.2.4. *The long and short factors of G and G^* are related by:*

$$\widehat{\pi}^L_m(G^*) = \Xi\widehat{\pi}^S_m(G)$$

Proof. The action of ε_G takes $r \in \mathbf{R}^{\mathbf{S}}_{\mathbf{G}}$ to some $r' \in \mathbf{R}^{\mathbf{L}}_{\mathbf{G}^*}$. It is easily checked that $\xi_G X^{v(r)} = X^{v(r')}$ and thus the claim follows $\square$

Remark. Although the notations are similar, from context it is clear that $X^{v(r)} \in \mathfrak{E}(G)$ while $X^{v(r')} \in \mathfrak{E}(G^*)$.

Although they seem new, we have already seen some examples of the long-short factorizations in Theorem 3.1.2. One has:

Theorem 3.2.5. *If G is nonsimply-laced and q its characteristic, then:*

$$\widehat{\pi}^S_q(G) = \widehat{\pi}_{\epsilon^*[\rho]}(G)$$

In particular, $\widehat{\pi}^S_q(G)$ is the common factor appearing in the exotic factorizations of Theorem 3.1.2.

Proof. Since q is prime, factorization of $\widehat{\pi}^S_q(G)$ in $\mathfrak{E}(G)$ is given by:

$$\widehat{\pi}^S_q(G) = \prod_{r \in \mathbf{R}^+_{\mathbf{S}}} \Phi_q(X^{v(r)}, X^{-v(r)})$$

Since $W(G)$ acts transitively on the short roots, no subset of these factors is a $W(G)$-symmetric set. Hence $\widehat{\pi}^S_q(G)$ is irreducible in $\mathfrak{R}(G)$. Similarly, $\Xi\widehat{\pi}^S_q(G^*)$ must be irreducible in $\mathfrak{R}(G)$.

Now consider the exotic factorization and ^{LS}-factorization of $\widehat{\pi}_{q[\rho]}(G) = \widehat{\pi}_{\epsilon^*(\epsilon[\rho])}(G)$ in $\mathfrak{R}(G)$:

$$\widehat{\pi}_{\epsilon^*(\epsilon[\rho])}(G) = \widehat{\pi}_{\epsilon^*[\rho]}(G) \cdot \Xi^* \widehat{\pi}_{\epsilon[\rho]}(G^*)$$

$$\widehat{\pi}_{q[\rho]}(G) = \widehat{\pi}^L_q(G) \cdot \widehat{\pi}^S_q(G)$$

Thus $\widehat{\pi}_{\epsilon^*[\rho]}(G)$ and $\Xi^* \widehat{\pi}_{\epsilon[\rho]}(G^*)$ equal $\widehat{\pi}^L_q(G)$ and $\widehat{\pi}^S_q(G)$ in some order. By Theorem 3.2.4, the factor $\widehat{\pi}^L_q(G)$ equals $\Xi^* \widehat{\pi}^S_q(G^*)$. The $\mathfrak{C}(G)$-factors of $\widehat{\chi}^S_q(G)$ all come from short roots of G by the definition of the short factor. But the short roots of G are not in the image of ε^* acting on the root lattice of G^*, so $\widehat{\pi}^S_q(G)$ cannot be in the image of Ξ^*. Thus $\widehat{\pi}^S_q(G) = \widehat{\pi}_{\epsilon^*[\rho]}(G)$ as claimed. □

Remark. In general, one has that for $m \geq 2$, $\widehat{\pi}^S_m(G)$ is an irrep only in the cases given in 3.2.5 (for $m = 1$ this is trivial since $\widehat{\pi}^L_1(G) = \widehat{\pi}^S_1(G) = 1$). $\widehat{\pi}^S_m(G)$ is always virtual if m is larger than the characteristic of G and in the sole remaining case $\widehat{\pi}^S_2(\mathcal{G}_2)$ is a reducible honest representation. For $m \geq 2$, $\widehat{\pi}^L_m(G)$ is always virtual.

The dimensions of the long and short factors of $\widehat{\pi}_{m[\rho]}(G)$ are easily obtained from their defintion:

Theorem 3.2.6. *The virtual dimension of $\widehat{\pi}^L_m(G)$ is $m^{|\mathbf{R}^{\mathbf{L}}_{\mathbf{G}}|}$ and similarly the virtual dimension of $\widehat{\pi}^S_m(G)$ is $m^{|\mathbf{R}^{\mathbf{S}}_{\mathbf{G}}|}$.*

Proof. Combining the Weyl Character Formula with Theorem 3.2.1, in $\mathfrak{C}(G)$ the long factor $\widehat{\pi}^L_m(G)$ splits into $|\mathbf{R}^{\mathbf{L}}_{\mathbf{G}}|$ factors of the form:

$$\left(X^{(m-1)\cdot v(r)} + X^{(m-3)\cdot v(r)} + \ldots + X^{-(m-3)\cdot v(r)} + X^{-(m-1)\cdot v(r)}\right)$$

Each such factor has dimension m by specialization at $\{X_i = 1\}_{i=1}^n$ and so the dimension of the long factor follows. An analogous argument applies for the dimension of the short factor. $\square$

Since they are virtual representations, one would like to know the decomposition into irreducible representations of both long and short factors. A priori there is no reason to expect that the number of summands in this decomposition should be finite or bounded (since the dimensions grow without bound), but nevertheless we start by looking at the long and short factors of $\pi_{m[\rho]}(\mathcal{G}_2)$ to see what can be said about their decomposition into irreducibles. For reason of convenience in later discussion, we present the results using unshifted indices in Table 3.2.

Note that there is a regular pattern among all the long factors and among the short factors in the last three rows of Table 3.2. In fact the short factors in the first two rows also follow the same pattern since for $m = 1$, $\pi_{[-2,1]}(\mathcal{G}_2) = 0$ and $\pi_{[-2,0]}(\mathcal{G}_2) = \pi_{[0,0]}(\mathcal{G}_2)$ while for $m = 2$, $\pi_{[-1,1]}(\mathcal{G}_2) = \pi_{[-1,0]}(\mathcal{G}_2) = 0$. A quick calculation with the Weyl Dimension formula with the apparent pattern shows that the virtual dimensions of $\pi_{[0,m]}(\mathcal{G}_2) \ominus \pi_{[1,m-1]}(\mathcal{G}_2) \oplus \pi_{[0,m-1]}(\mathcal{G}_2)$ and $\pi_{[m,0]}(\mathcal{G}_2) \ominus \pi_{[m-3,1]}(\mathcal{G}_2) \oplus \pi_{[m-3,0]}(\mathcal{G}_2)$ for arbitrary m are both $(m+1)^3$ which are the dimensions predicted by Theorem 3.2.6, so we are led to the following result for $\mathcal{G}_2$:

m	$\pi^L_m(\mathcal{G}_2)$	$\pi^S_m(\mathcal{G}_2)$
1	$\pi_{[0,1]}(\mathcal{G}_2) \ominus \pi_{[1,0]}(\mathcal{G}_2) \oplus \pi_{[0,0]}(\mathcal{G}_2)$	$\pi_{[1,0]}(\mathcal{G}_2) \oplus \pi_{[0,0]}(\mathcal{G}_2)$
2	$\pi_{[0,2]}(\mathcal{G}_2) \ominus \pi_{[1,1]}(\mathcal{G}_2) \oplus \pi_{[0,1]}(\mathcal{G}_2)$	$\pi_{[2,0]}(\mathcal{G}_2)$
3	$\pi_{[0,3]}(\mathcal{G}_2) \ominus \pi_{[1,2]}(\mathcal{G}_2) \oplus \pi_{[0,2]}(\mathcal{G}_2)$	$\pi_{[3,0]}(\mathcal{G}_2) \ominus \pi_{[0,1]}(\mathcal{G}_2) \oplus \pi_{[0,0]}(\mathcal{G}_2)$
4	$\pi_{[0,4]}(\mathcal{G}_2) \ominus \pi_{[1,3]}(\mathcal{G}_2) \oplus \pi_{[0,3]}(\mathcal{G}_2)$	$\pi_{[4,0]}(\mathcal{G}_2) \ominus \pi_{[1,1]}(\mathcal{G}_2) \oplus \pi_{[1,0]}(\mathcal{G}_2)$
5	$\pi_{[0,5]}(\mathcal{G}_2) \ominus \pi_{[1,4]}(\mathcal{G}_2) \oplus \pi_{[0,4]}(\mathcal{G}_2)$	$\pi_{[5,0]}(\mathcal{G}_2) \ominus \pi_{[2,1]}(\mathcal{G}_2) \oplus \pi_{[2,0]}(\mathcal{G}_2)$
...	...	...

Table 3.2: Long and Short Factors of $\pi_{[m,m]}(\mathcal{G}_2)$

Theorem 3.2.7. *In $\mathfrak{R}(\mathcal{G}_2)$, the long and short factors of $\pi_{[m,m]}(\mathcal{G}_2)$ are as follows:*

$$\pi^L_m(\mathcal{G}_2) = \pi_{[0,m]}(\mathcal{G}_2) \ominus \pi_{[1,m-1]}(\mathcal{G}_2) \oplus \pi_{[0,m-1]}(\mathcal{G}_2) \tag{3.2.5}$$

$$\pi^S_m(\mathcal{G}_2) = \pi_{[m,0]}(\mathcal{G}_2) \ominus \pi_{[m-3,1]}(\mathcal{G}_2) \oplus \pi_{[m-3,0]}(\mathcal{G}_2) \tag{3.2.6}$$

Proof. This is proven by a variant of the proof of Algorithm 1.1.2. Replace each representation in (3.2.5) with the corresponding characters and write the characters in terms of the various $E_{[I]}(\mathcal{G}_2)$ and $E^L_{[I]}(\mathcal{G}_2)$. Clearing denominators in the resulting expression, (3.2.5) therefore becomes:

$$E^L_{(m+1)[\rho]}(\mathcal{G}_2) \cdot E_{[\rho]}(\mathcal{G}_2) = E^L_{[\rho]}(\mathcal{G}_2) \cdot \big(E_{[1,m+1]}(\mathcal{G}_2) - E_{[2,m]}(\mathcal{G}_2) + E_{[1,m]}(\mathcal{G}_2)\big)$$

When expanded out, the LHS contains 72 terms while the right side contains 216 terms; it is then straightforward to check that the necessary terms cancel to give equality of the two sides. The proof of (3.2.6) is analogous. □

Remark. Theorem 3.2.7 implies $\pi_m^S(\mathcal{G}_2)$ is an honest representation of $\mathcal{G}_2$ iff $m = 0, 1$, or 2 and $\pi_m^L(\mathcal{G}_2)$ is honest iff $m = 0$ (in which case it is trivial), thus verifying the remark after Theorem 3.2.5 in the case of $\mathcal{G}_2$. For $m = 0, 2$, $\pi_m^S(\mathcal{G}_2)$ is irreducible, but for $m = 1$ it is not. Thus $\pi_{[1,1]}(\mathcal{G}_2)$ gives an example of an irreducible representation which has an honest reducible factor in $\mathfrak{R}(\mathcal{G}_2)$. Indeed it appears that this phenomenon is unique:

Conjecture 3.2.1. *Let $[I]$ be a dominant weight of G and suppose $\pi_{[I]}(G)$ factors in $\mathfrak{R}(G)$ such that one of the factors is an honest representation which is not irreducible. Then $G = \mathcal{G}_2$ and $[I] = [1, 1]$.*

Another consequence of the truth of Theorem 3.2.7 is that for $m \geq 3$, the irreducible representation $\pi_{[m,m]}(\mathcal{G}_2)$ factors into two purely virtual representations; contrast this with the factorizations of Theorems 3.1.1 and 3.1.2 in which one of the factors is always honest and irreducible.

For the other nonsimply-laced groups of rank ≤ 4, basic calculations lead to the following results:

Theorem 3.2.8. *The long and short factors of $\pi_{m[\rho]}(G)$ in $\mathfrak{R}(G)$ for the other groups of rank ≤ 4 are as follows:*

$$\pi_m^L(\mathcal{B}_2) = \pi_{[m,0]}(\mathcal{B}_2) \ominus \pi_{[m-1,0]}(\mathcal{B}_2)$$

$$\pi_m^S(\mathcal{B}_2) = \pi_{[0,m]}(\mathcal{B}_2) \ominus \pi_{[0,m-2]}(\mathcal{B}_2)$$

$\pi_m^L(\mathcal{B}_3)$		$\pi_m^S(\mathcal{B}_3)$	
$\oplus\pi_{[m,m,0]}(\mathcal{B}_3)$	$\ominus\pi_{[m+1,m-1,0]}(\mathcal{B}_3)$	$\oplus\pi_{[0,0,m]}(\mathcal{B}_3)$	$\ominus\pi_{[1,0,m-2]}(\mathcal{B}_3)$
$\oplus\pi_{[m,m-1,0]}(\mathcal{B}_3)$	$\ominus\pi_{[m-1,m,0]}(\mathcal{B}_3)$	$\oplus\pi_{[0,1,m-4]}(\mathcal{B}_3)$	$\ominus\pi_{[0,0,m-4]}(\mathcal{B}_3)$

$\pi_m^L(\mathcal{C}_3)$		$\pi_m^S(\mathcal{C}_3)$	
$\oplus\pi_{[0,0,m]}(\mathcal{C}_3)$	$\ominus\pi_{[1,0,m-1]}(\mathcal{C}_3)$	$\oplus\pi_{[m,m,0]}(\mathcal{C}_3)$	$\ominus\pi_{[m+2,m-2,0]}(\mathcal{C}_3)$
$\oplus\pi_{[0,1,m-2]}(\mathcal{C}_3)$	$\ominus\pi_{[0,0,m-2]}(\mathcal{C}_3)$	$\oplus\pi_{[m,m-2,0]}(\mathcal{C}_3)$	$\ominus\pi_{[m-2,m,0]}(\mathcal{C}_3)$

$\pi_m^L(\mathcal{B}_4)$		$\pi_m^S(\mathcal{B}_4)$	
$\oplus\pi_{[m,m,m,0]}(\mathcal{B}_4)$	$\ominus\pi_{[m,m+1,m-1,0]}(\mathcal{B}_4)$	$\oplus\pi_{[0,0,0,m]}(\mathcal{B}_4)$	$\ominus\pi_{[0,1,0,m-2]}(\mathcal{B}_4)$
$\oplus\pi_{[m+1,m,m-1,0]}(\mathcal{B}_4)$	$\ominus\pi_{[m+1,m-1,m,0]}(\mathcal{B}_4)$	$\oplus\pi_{[1,0,1,m-4]}(\mathcal{B}_4)$	$\ominus\pi_{[0,0,2,m-6]}(\mathcal{B}_4)$
$\oplus\pi_{[m-1,m+1,m-1,0]}(\mathcal{B}_4)$	$\ominus\pi_{[m-1,m,m,0]}(\mathcal{B}_4)$	$\oplus\pi_{[1,0,1,m-6]}(\mathcal{B}_4)$	$\ominus\pi_{[2,0,0,m-4]}(\mathcal{B}_4)$
$\oplus\pi_{[m,m-1,m,0]}(\mathcal{B}_4)$	$\ominus\pi_{[m,m,m-1,0]}(\mathcal{B}_4)$	$\oplus\pi_{[0,0,0,m-6]}(\mathcal{B}_4)$	$\ominus\pi_{[0,1,0,m-6]}(\mathcal{B}_4)$

$\pi_m^L(\mathcal{C}_4)$		$\pi_m^S(\mathcal{C}_4)$	
$\oplus\pi_{[0,0,0,m]}(\mathcal{C}_4)$	$\ominus\pi_{[0,1,0,m-1]}(\mathcal{C}_4)$	$\oplus\pi_{[m,m,m,0]}(\mathcal{C}_4)$	$\ominus\pi_{[m,m+2,m-2,0]}(\mathcal{C}_4)$
$\oplus\pi_{[1,0,1,m-2]}(\mathcal{C}_4)$	$\ominus\pi_{[0,0,2,m-3]}(\mathcal{C}_4)$	$\oplus\pi_{[m+2,m,m-2,0]}(\mathcal{C}_4)$	$\ominus\pi_{[m+2,m-2,m,0]}(\mathcal{C}_4)$
$\oplus\pi_{[1,0,1,m-3]}(\mathcal{C}_4)$	$\ominus\pi_{[2,0,0,m-2]}(\mathcal{C}_4)$	$\oplus\pi_{[m-2,m+2,m-2,0]}(\mathcal{C}_4)$	$\ominus\pi_{[m-2,m,m,0]}(\mathcal{C}_4)$
$\oplus\pi_{[0,0,0,m-3]}(\mathcal{C}_4)$	$\ominus\pi_{[0,1,0,m-3]}(\mathcal{C}_4)$	$\oplus\pi_{[m,m-2,m,0]}(\mathcal{C}_4)$	$\ominus\pi_{[m,m,m-2,0]}(\mathcal{C}_4)$

$\pi_m^L(\mathcal{F}_4)$		$\pi_m^S(\mathcal{F}_4)$	
$\oplus\pi_{[0,0,m,m]}(\mathcal{F}_4)$	$\ominus\pi_{[1,0,m-1,m+1]}(\mathcal{F}_4)$	$\oplus\pi_{[m,m,0,0]}(\mathcal{F}_4)$	$\ominus\pi_{[m+2,m-2,0,1]}(\mathcal{F}_4)$
$\oplus\pi_{[0,1,m-2,m+2]}(\mathcal{F}_4)$	$\ominus\pi_{[1,0,m,m-1]}(\mathcal{F}_4)$	$\oplus\pi_{[m+4,m-4,1,0]}(\mathcal{F}_4)$	$\ominus\pi_{[m-2,m,0,1]}(\mathcal{F}_4)$
$\oplus\pi_{[2,0,m-1,m]}(\mathcal{F}_4)$	$\ominus\pi_{[0,0,m-2,m+3]}(\mathcal{F}_4)$	$\oplus\pi_{[m,m-2,0,2]}(\mathcal{F}_4)$	$\ominus\pi_{[m+6,m-4,0,0]}(\mathcal{F}_4)$
$\oplus\pi_{[0,1,m,m-2]}(\mathcal{F}_4)$	$\ominus\pi_{[0,2,m-3,m+2]}(\mathcal{F}_4)$	$\oplus\pi_{[m-4,m,1,0]}(\mathcal{F}_4)$	$\ominus\pi_{[m+4,m-6,2,0]}(\mathcal{F}_4)$
$\oplus\pi_{[0,1,m-3,m+3]}(\mathcal{F}_4)$	$\ominus\pi_{[2,1,m-2,m]}(\mathcal{F}_4)$	$\oplus\pi_{[m+6,m-6,1,0]}(\mathcal{F}_4)$	$\ominus\pi_{[m,m-4,1,2]}(\mathcal{F}_4)$
$\oplus\pi_{[1,2,m-3,m+1]}(\mathcal{F}_4)$	$\ominus\pi_{[0,0,m+1,m-3]}(\mathcal{F}_4)$	$\oplus\pi_{[m+2,m-6,2,1]}(\mathcal{F}_4)$	$\ominus\pi_{[m-6,m+2,0,0]}(\mathcal{F}_4)$
$\oplus\pi_{[1,2,m-2,m-1]}(\mathcal{F}_4)$	$\ominus\pi_{[0,2,m-1,m-2]}(\mathcal{F}_4)$	$\oplus\pi_{[m-2,m-4,2,1]}(\mathcal{F}_4)$	$\ominus\pi_{[m-4,m-2,2,0]}(\mathcal{F}_4)$
$\oplus\pi_{[0,1,m,m-3]}(\mathcal{F}_4)$	$\ominus\pi_{[0,3,m-3,m]}(\mathcal{F}_4)$	$\oplus\pi_{[m-6,m,1,0]}(\mathcal{F}_4)$	$\ominus\pi_{[m,m-6,3,0]}(\mathcal{F}_4)$
$\oplus\pi_{[3,0,m-2,m]}(\mathcal{F}_4)$	$\ominus\pi_{[1,0,m-3,m+3]}(\mathcal{F}_4)$	$\oplus\pi_{[m,m-4,0,3]}(\mathcal{F}_4)$	$\ominus\pi_{[m+6,m-6,0,1]}(\mathcal{F}_4)$
$\oplus\pi_{[2,0,m-3,m+2]}(\mathcal{F}_4)$	$\ominus\pi_{[2,1,m-3,m+1]}(\mathcal{F}_4)$	$\oplus\pi_{[m+4,m-6,0,2]}(\mathcal{F}_4)$	$\ominus\pi_{[m+2,m-6,1,2]}(\mathcal{F}_4)$
$\oplus\pi_{[1,2,m-3,m]}(\mathcal{F}_4)$	$\ominus\pi_{[2,1,m-2,m-1]}(\mathcal{F}_4)$	$\oplus\pi_{[m,m-6,2,1]}(\mathcal{F}_4)$	$\ominus\pi_{[m-2,m-4,1,2]}(\mathcal{F}_4)$
$\oplus\pi_{[2,0,m-1,m-2]}(\mathcal{F}_4)$	$\ominus\pi_{[1,0,m,m-3]}(\mathcal{F}_4)$	$\oplus\pi_{[m-4,m-2,0,2]}(\mathcal{F}_4)$	$\ominus\pi_{[m-6,m,0,1]}(\mathcal{F}_4)$
$\oplus\pi_{[0,0,m-3,m+3]}(\mathcal{F}_4)$	$\ominus\pi_{[1,0,m-3,m+2]}(\mathcal{F}_4)$	$\oplus\pi_{[m+6,m-6,0,0]}(\mathcal{F}_4)$	$\ominus\pi_{[m+4,m-6,0,1]}(\mathcal{F}_4)$
$\oplus\pi_{[0,0,m,m-3]}(\mathcal{F}_4)$	$\ominus\pi_{[0,2,m-3,m]}(\mathcal{F}_4)$	$\oplus\pi_{[m-6,m,0,0]}(\mathcal{F}_4)$	$\ominus\pi_{[m,m-6,2,0]}(\mathcal{F}_4)$
$\oplus\pi_{[0,1,m-3,m+1]}(\mathcal{F}_4)$	$\ominus\pi_{[1,0,m-1,m-2]}(\mathcal{F}_4)$	$\oplus\pi_{[m+2,m-6,1,0]}(\mathcal{F}_4)$	$\ominus\pi_{[m-4,m-2,0,1]}(\mathcal{F}_4)$
$\oplus\pi_{[0,1,m-2,m-1]}(\mathcal{F}_4)$	$\ominus\pi_{[0,0,m-2,m]}(\mathcal{F}_4)$	$\oplus\pi_{[m-2,m-4,1,0]}(\mathcal{F}_4)$	$\ominus\pi_{[m,m-4,0,0]}(\mathcal{F}_4)$

Proof. Analogous to proof of Theorem 3.2.7. $\square$

Remark. Note that as G grows, the number of terms on each side grows rapidly and these proofs quickly become unfeasible to do by hand. For example the LHS for either factor of $\mathcal{F}_4$ has 221184 terms while the RHS has over 7 million terms; as a result all calculations were done in MAPLE ([M13] for $\mathcal{F}_4$, [M11] for all others). Using the Weyl Dimension Formula, it is easily checked that the dimensions of the summands add up to the predicted dimensions of the long and short factors in each case; this is a simple way to verify that the MAPLE calculations of the summands are likely correct.

Looking through these results, we see several patterns. First note that for $\mathcal{G}_2$ there are 3^1 summands in both the long and short factors; for $\mathcal{B}_n$ and $\mathcal{C}_n$ ($n = 2, 3, 4$), there are 2^{n-1} summands, and for $\mathcal{F}_4$ there are 2^5 summands; here we have highlighted the relationship between the number of summands and the characteristic of the respective group. Among the $\mathcal{B}_n$, $\mathcal{C}_n$, and $\mathcal{F}_4$ cases the summands are evenly split between $\oplus$ and $\ominus$ terms. These observations lead to the following conjecture for $\mathcal{B}_n$ and $\mathcal{C}_n$ in general:

Conjecture 3.2.2. *The long and short factors of* $\pi_{m[\rho]}(\mathcal{B}_n)$ *and* $\pi_{m[\rho]}(\mathcal{C}_n)$ *are always a sum of* 2^{n-1} *irreducible summands. The signs of these irreducible summands in the decomposition are evenly split between* $\oplus$ *and* $\ominus$.

Another pattern which is not as apparent as these occurs among the highest weights of the various summands. Based on our observations in Theorems 3.2.7 and 3.2.8 we make the following guess about the behaviors of the summands of these factors in general:

Conjecture 3.2.3. *For each summand of $\pi_m^L(G)$, there is a corresponding summand of $\pi_m^S(G^*)$ with the same sign. The highest weights of corresponding summands are related as follows:*

G	Highest Weight of Summand of $\pi_m^L(G)$	Highest Weight of Corresponding Summand of $\pi_m^S(G^*)$
$\mathcal{B}_n$	$[m+k_1, m+k_2, \dots, m+k_{n-1}, k_n]$	$[m+2k_1, m+2k_2, \dots, m+2k_{n-1}, k_n]$
$\mathcal{C}_n$	$[k_1, k_2, \dots, k_{n-1}, m+k_n]$	$[k_1, k_2, \dots, k_{n-1}, m+2k_n]$
$\mathcal{F}_4$	$[k_1, k_2, m+k_3, m+k_4]$	$[m+2k_4, m+2k_3, k_2, k_1]$
$\mathcal{G}_2$	$[k_1, m+k_2]$	$[m+3k_2, k_1]$

Remark. In Theorems 3.2.7 and 3.2.8 we have placed these corresponding summands in corresponding positions to make this pattern easier to see.

Note that if we focus on the m-independent portion of the weights, then ϵ_G applied to the highest weight of a summand of $\pi_{m[\rho]}^L(G)$ gives the highest weight of the corresponding summand of $\pi_{m[\rho]}^S(G^*)$ while the m-dependent portion of both weights is unchanged. That such a relationship exists is somewhat surprising, but not entirely unexpected since the roots which give rise to the long and short factors are also themselves related by ε_G. It would be interesting to obtain a general description of which weights $[k_1, \dots, k_n]$ appear in the $\mathcal{B}_n$ and $\mathcal{C}_n$ cases in general, especially if this description could also be applied to $\mathcal{G}_2$ and $\mathcal{F}_4$ to produce the weight sets already calculated above.

3.2.3 Γ^{LS}-factorizations

In this section we combine our observations from §3.2.1 and §3.2.2 to deduce the existence of yet another class of factors in $\mathfrak{R}(G)$ when G is nonsimply-laced.

Recall that in $\mathfrak{E}(G)$ the character $\gamma_d(G)$ splits as:

$$\gamma_d(G) = \prod_{r \in \mathbf{R}_\mathbf{G}^+} \Phi_d(X^{v(r)}, X^{-v(r)})$$

As we remarked in §3.2.2, when G is nonsimply-laced, the $W(G)$-action on the roots breaks up into separate actions on long and short roots. As a consequence, in the nonsimply-laced case we can rewrite the above factorization of $\gamma_d(G)$ in the form of an ^{LS}-factorization:

$$\begin{aligned}\gamma_d(G) &= \left[\prod_{r \in \mathbf{R}_\mathbf{G}^\mathbf{L}} \Phi_d(X^{v(r)}, X^{-v(r)})\right] \cdot \left[\prod_{r \in \mathbf{R}_\mathbf{G}^\mathbf{S}} \Phi_d(X^{v(r)}, X^{-v(r)})\right] \\ &= \gamma_d^L(G) \cdot \gamma_d^S(G)\end{aligned}$$

Each of $\gamma_d^L(G)$ and $\gamma_d^S(G)$ is clearly $W(G)$-symmetric and so they are characters of long and short type factors of $\Gamma_d(G)$ in $\mathfrak{R}(G)$:

$$\Gamma_d(G) = \Gamma_d^L(G) \cdot \Gamma_d^S(G)$$

The existence of the Γ^{LS}-factors ensures that there are no conflicts between the Γ-factorizations and the ^{LS}-factorizations with regards to unique factorization. As they behave very much like these previously studied examples, we will only briefely summarize some of their properties; these properties blend some of our previous theorems and results concerning the Γ-factorizations and ^{LS}-factorizations.

Theorem 3.2.9. *The Γ^{LS}-factors satisfy the following relationships to one another and to the other types of factorizations already discussed:*

$$\widehat{\pi}^{L}_{m[\rho]}(G) = \prod_{d|m} \Gamma^{L}_{d}(G)$$

$$\widehat{\pi}^{S}_{m[\rho]}(G) = \prod_{d|m} \Gamma^{S}_{d}(G)$$

$$\Gamma^{L}_{d}(G^{*}) = \Xi\Gamma^{S}_{d}(G)$$

Analogous to the remark after Theorem 3.2.2, the virtual dimensions of the Γ^{LS}-factors are given by:

$$\dim(\Gamma^{L}_{d}(G)) = \begin{cases} p^{|\mathbf{R}^{\mathbf{L}}_{\mathbf{G}}|} & d = p^{k} \text{ with } p \text{ prime} \\ 1 & \text{otherwise} \end{cases}$$

$$\dim(\Gamma^{S}_{d}(G)) = \begin{cases} p^{|\mathbf{R}^{\mathbf{S}}_{\mathbf{G}}|} & d = p^{k} \text{ with } p \text{ prime} \\ 1 & \text{otherwise} \end{cases}$$

Finally, for p prime and $k \geq 1$ one has:

$$\Gamma^{L}_{p^{k}}(G) = \Psi^{p^{k-1}}\Gamma^{L}_{p}(G)$$

$$\Gamma^{S}_{p^{k}}(G) = \Psi^{p^{k-1}}\Gamma^{S}_{p}(G)$$

Proof. The proofs are similar to the corresponding statements we have already proved about the Γ-factorizations and ^{LS}-factorizations and thus are omitted. $\square$

Chapter 4

Related Results and Further Directions of Research

Within the course of our work, there have been several open questions which have yet to be resolved. In addition, many of our results clearly apply to more general situations. For example, there are analogues of the Weyl Character Formula, root systems, Weyl groups, etc. for the class of generalized Kac-Moody algebras and their corresponding Lie groups; the Lie algebras of the compact semisimple Lie groups are special cases of such generalized Kac-Moody algebras. Consequently, there should be analogues of most of our results for the more general setting of generalized Kac-Moody algebras and their corresponding Lie groups. Here we discuss the potential for further research arising from our work and connect our results to questions of divisibility in other settings.

4.1 The Main Unsettled Questions

Through the course of this dissertation, we have left several important questions unanswered. Some, such as Conjectures 3.2.2 and 3.2.3 should be relatively easy to settle in the affirmative. Conjecture 3.2.1 seems more difficult to prove at this time, but nevertheless seems tractable. The big unsettled questions arising from our work which we would like to know the answers to are the following:

Conjecture 4.1.1. *Among the irreps of arbitrary G, all series factorizations are accounted for by the types already discussed.*

Conjecture 4.1.2. *If an irrep factors in $\mathfrak{R}(G)$, then its factors are all accounted for by one or more of the factorization series already discussed. In particular there are no sporadic factorizations appearing among the irreps of any G.*

Although simple to state, these conjectures seem to admit no easy proofs; rather it seems that one needs some rigid results on the possible structure of potential factors as well as some results on where these factors may lie in order to completely settle them.

4.2 Divisibility Properties of Recursively Defined Integer Sequences

In light of the divisibility properties we have observed among the characters, we now examine the corresponding divisibility statements when the characters are forced to be integers. In this case we are therefore interested in conjugacy classes c_G of G such that $\chi_{[I]}$ evaluated on c_G is integer-valued for all $[I]$.

Definition 4.2.1. *A* $\mathbb{Z} -$ **class** *of* G *is a conjugacy class* $c_G \subset G$ *such that* $\chi_{[I]}(g) \in \mathbb{Z}$ *for* $g \in c_G$.

We will start in the case of $\mathcal{A}_1$ where the Klimyk relationship between characters took the following form:

$$\chi_{[n+1]}(\mathcal{A}_1) = \chi_{[1]}(\mathcal{A}_1)\chi_{[n]}(\mathcal{A}_1) - \chi_{[n-1]}(\mathcal{A}_1)$$

Since $\chi_{[0]}(\mathcal{A}_1) = 1$ and $\chi_{[-1]}(\mathcal{A}_1) = 0$, it is clear that a class $c_{\mathcal{A}_1}$ is a $\mathbb{Z}$-class iff the fundamental character $\chi_{[1]}$ evaluated on $c_{\mathcal{A}_1}$ is an integer. As the coordinate X_1 appearing in the character is a $U(1)$-valued functions, this leaves (up to complex conjugation) 5 choices for X_1: $X_1 = \zeta_1,\ \zeta_2,\ \zeta_3,\ \zeta_4$, or ζ_6 where ζ_j is a primitive j^{th} root of unity. Each X_i represents a conjugacy class of $SU(2)$, and plugging in these 5 values gives Table 4.1 summarizing the values of the characters on these conjugacy classes.

$\widehat{\chi}_{[n]} \backslash X_1$	ζ_2	ζ_3	ζ_4	ζ_6	ζ_1
$\widehat{\chi}_{[1]}$	1	1	1	1	1
$\widehat{\chi}_{[2]}$	-2	-1	0	1	2
$\widehat{\chi}_{[3]}$	3	0	-1	0	3
$\widehat{\chi}_{[4]}$	-4	1	0	-1	4
$\widehat{\chi}_{[5]}$	5	-1	1	-1	5
$\widehat{\chi}_{[6]}$	-6	0	0	0	6
$\widehat{\chi}_{[7]}$	7	1	-1	1	7
$\widehat{\chi}_{[8]}$	-8	-1	0	1	8
$\widehat{\chi}_{[9]}$	9	0	1	0	9
$\widehat{\chi}_{[10]}$	-10	1	0	-1	10
$\widehat{\chi}_{[11]}$	11	-1	-1	-1	11
$\widehat{\chi}_{[12]}$	-12	0	0	0	12
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 4.1: Integer-Valued Irreducible Character Sequences of $\mathbb{Z}$-classes in $\mathcal{A}_1$

Definition 4.2.2. *A sequence* $\{S_j\}_{j=1}^{\infty}$ *of integers is a* **divisible sequence** *if for all* $1 \leq k \leq \ell$ *one has:*

$$k|\ell \quad \Longrightarrow \quad S_k|S_\ell \tag{4.2.1}$$

Simple inspection shows that the sequences in Table 4.1 are all divisible sequences.

On the other hand, none of our proofs of the factorizations of characters and representations relied explicitly on the fact that the X_i are $U(1)$-valued. Thus one can extend our divisibility results to consider 'virtual classes' of G which are defined as follows:

Definition 4.2.3. *A class c_G is a* **virtual class** *of G such that some or all of the X_i are not $U(1)$-valued*

Clearly virtual classes do not exist in G itself; however they can be interpreted as conjugacy classes in certain noncompact forms of G; thus we refer to them as 'virtual' only inasmuch as they do not correspond to any class in the compact group G.

We will use the notation $c(y_1, \ldots, y_n)$ to refer to a (possibly virtual) class of G such that the evaluation of χ_{ω_j} on $c(y_1, \ldots, y_n)$ equals y_j. Thus the y_i implicitly determine the values of $X_1, \ldots, X_n$ describing the class.

Since our interest is in divisible sequences, we will focus on $\mathbb{Z}$-classes of G; such classes are completely determined by the following result:

Theorem 4.2.1. *A necessary and sufficient condition that a (possibly virtual) class $c(y_1, \ldots, y_n)$ is a $\mathbb{Z}$-class of G is that all y_i are integers.*

Proof. That all y_i be integers is clearly necessary since $y_i = \chi_{\omega_i}(G; c(y_1, \ldots, y_n))$.

On the other hand, since $\chi_{[I]}$ is a $\mathbb{Z}$-polynomial in terms of the fundamental characters, the given condition is also sufficient. $\square$

In the case of $\mathcal{A}_1$, a simple example of a divisible sequence arising from a virtual class is the sequence $\{\widehat{\chi}_{[n]}(\mathcal{A}_1; c(3))\}_{n=1}^{\infty}$. The Klimyk relation implies the sequence of character values of this class satisfy the following linear recurrence:

$$\widehat{\chi}_{[n+1]}(\mathcal{A}_1; c(3)) = 3\widehat{\chi}_{[n]}(\mathcal{A}_1; c(3)) - \widehat{\chi}_{[n-1]}(\mathcal{A}_1; c(3))$$

This gives the sequence $\{1, 3, 8, 21, 55, \ldots\}$. For $n = 1 \ldots 5$ we see these are even-index Fibonacci numbers: $\widehat{\chi}_{[n]}(\mathcal{A}_1; c(3)) = F_{2n}$. It is easy to check that the even-index Fibonacci numbers satisfy the same recurrence relation, so the two sequences are indeed the same.

As a result, we thus have two proofs that the sequence $\{\widehat{\chi}_{[n]}(\mathcal{A}_1; 3)\}_{n=1}^{\infty}$ forms a divisibility sequence. On the one hand, it is well known that the Fibonacci numbers form a divisible sequence; since the sequence $\{\widehat{chi}_{[n]}(\mathcal{A}_1; c(3))\}_{n=1}^{\infty}$ is the subsequence of the Fibonacci numbers of even-index, they inherit the property of being a divisible sequence from the Fibonacci numbers. On the other hand, Theorem 3.1.1 shows that the sequence of Laurent polynomials $\{\widehat{\chi}_{[n]}(\mathcal{A}_1; c(z))\}_{n=1}^{\infty}$ has the divisibility property for a variable z, so that whenever we evaluate the $\widehat{\chi}_{[n]}$ on a $\mathbb{Z}$-class the resulting sequence is a divisible sequence; in the above example one has $X_1 = \frac{3+\sqrt{5}}{2}$.

For $y_1 = 4, 5, 6, 7, \ldots$ one obtains several other interesting recursively defined divisible sequences which are given in Table 4.2.

v_1	Sequence $\{\widehat{\chi}_{[n]}(\mathcal{A}_1; c(y_1))\}_{n=1}^{\infty}$	OEIS Number
4	$1, 4, 15, 56, 209, 780, 2911, 10864, \ldots$	A001353
5	$1, 5, 24, 115, 551, 2640, 12649, 60605, \ldots$	A004254
6	$1, 6, 35, 204, 1189, 6930, 40391, 235416, \ldots$	A001109
7	$1, 7, 48, 329, 2255, 15456, 105937, 726103, \ldots$	A004187

Table 4.2: Some Divisible Sequences Arising from Virtual $\mathcal{A}_1$-Classes

These examples of virtual classes of $\mathcal{A}_1$ appear in $SL_2(\mathbb{C})$ and give rise to many 2^{nd} order linear recursive divisible sequences in their characters. However, it is not hard to see that not all 2^{nd} order linear recursive divisible sequences arise from such virtual classes. Most notably, the Fibonacci sequence does not arise from any class in $SL_2(\mathbb{C})$, even though its even-index subsequence does arise in this way as already noted.

However, if one considers classes in $GL_2(\mathbb{C})$ then it is possible to obtain all 2^{nd} order linear recursive sequences by choosing classes whose eigenvalues in the fundamental representation are certain algebraic numbers related to the recursive relationship satisfied by the sequence. See [BPP] for general details on the structure of linear recursive divisible sequences.

Next, suppose we look at the analogs of such divisible sequences arising from higher-rank Lie groups. Viewing the irreps of G as elements of $\mathfrak{R}(G)$, we let $y_1, \ldots, y_n$ be integers and let $S(y_1, \ldots, y_n)$ denote the array whose values are given by:

$$S_{i_1,\ldots,i_n}(y_1,\ldots,y_n) := \widehat{\chi}_{[I]}(G; c(y_1,\ldots,y_n)) \qquad\qquad [I] = [i_1,\ldots,i_n]$$

The existence of ordinary factorizations in $\mathfrak{R}(G)$ implies that this array satisfies the following analogue of (4.2.1):

$$i_k | j_k \;\forall\, 1 \le k \le n \quad \Longrightarrow \quad S_{i_1,\ldots,i_n}(G; c(y_1,\ldots,y_n)) | S_{j_1,\ldots,j_n}(G; c(y_1,\ldots,y_n))$$

This array is recursive due to the Klimyk relation on the character values, although the recursion is in general defined by several equations.

For example, looking at the virtual class $c(4,5)$ of $\mathcal{A}_2$ one obtains the 2-dimensional recursive divisible array of Table 4.3.

$i_2 \backslash i_1$	1	2	3	4	5	6	$\ldots$
1	1	4	11	25	49	82	$\ldots$
2	5	19	51	114	220	361	$\ldots$
3	21	79	211	470	904	1477	$\ldots$
4	86	323	862	1919	3689	6023	$\ldots$
5	351	1318	3517	7829	15049	24568	$\ldots$
6	1432	5377	14348	31939	61393	100225	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 4.3: Values of the Recursive Divisible Array $S_{i_1,i_2}(\mathcal{A}_2; c(4,5))$

Remark. By the reflexivity of the generators of $\mathcal{A}_1$ discussed in §2.1.1, the table of values of $S_{i_1,i_2}(\mathcal{A}_2; c(5,4))$ has the same values as Table 4.3 reflected across the main diagonal, and in general one has:

$$S_{i_1,i_2}(\mathcal{A}_2; c(p,q)) = S_{i_2,i_1}(\mathcal{A}_2; c(q,p))$$

The basic recursions in Table 4.3 are given by the following pair of equations, arising from the Klimyk relations on the fundamental irreducible representations:

$$S_{i_1+1,i_2}(\mathcal{A}_2; 4,5) = 4S_{i_1,i_2}(\mathcal{A}_2; c(4,5)) - S_{i_1-1,i_2+1}(\mathcal{A}_2; c(4,5)) - S_{i_1,i_2-1}(\mathcal{A}_2; c(4,5))$$

$$S_{i_1,i_2+1}(\mathcal{A}_2; 4,5) = 5S_{i_1,i_2}(\mathcal{A}_2; c(4,5)) - S_{i_1+1,i_2-1}(\mathcal{A}_2; c(4,5)) - S_{i_1-1,i_2}(\mathcal{A}_2; c(4,5))$$

Remark. The analogous basic recursions for arbitrary $S_{i_1,i_2}(\mathcal{A}_2; c(p,q))$ are obtained by replacing the 4 and 5 in the above recursions by p and q respectively.

If one wants to restrict to divisible subsequences of such arrays, then the obvious place to look is at the Γ-factorizations along the main diagonal in analogy with the sequences we constructed above from $\mathcal{A}_1$. Thus one defines:

$$S_i^{\Gamma}(G; c(y_1,\ldots,y_n)) := S_{i,\ldots,i}(G; c(y_1,\ldots,y_n))$$

From Theorem 3.2.1, these subsequences can be written in the following form:

$$S_i^{\Gamma}(G; c(y_1,\ldots,y_n)) = \prod_{k=1}^{|\mathbf{R}_G^+|} \left(\frac{\alpha_k^i - \beta_k^i}{\alpha_k - \beta_k} \right)$$

Here the α_k and β_k are algebraic numbers depending on G and the values of $y_1,\ldots,y_n$. Regardless of the values of the α_k and β_k, the general form that the sequence $S_i^{\Gamma}(G; c(y_1,\ldots,y_n))$ takes implies that not only is it divisible, but also that it satisfies a linear recurrence, see [BPP] for further details. The coefficients of this linear recurrence depend polynomially on the values of $y_1, \ldots, y_n$, these polynomials can be explicitly computed to lead to identities on the irreps of G.

As a simple example of using the linear recursions to find identities among the irreps, let us look at the sequences $S_i^\Gamma(\mathcal{A}_2; c(y_1, y_2))$. Some computation in MAPLE shows that for arbitrary y_1, y_2 they satisfy the following 6^{th} order recursion in general (for simplicity we abbreviate $S_i^\Gamma(\mathcal{A}_2; c(y_1, y_2))$ as S_i^Γ here):

$$\begin{aligned}
S_i^\Gamma =& (y_1y_2 - 3)S_{i-1}^\Gamma - (y_1^3 + y_2^3 - 5y_1y_2 + 6)S_{i-2}^\Gamma \\
&+ (y_1^2y_2^2 - 2y_1^3 - 2y_2^3 + 6y_1y_2 - 7)S_{i-3}^\Gamma - (y_1^3 + y_2^3 - 5y_1y_2 + 6)S_{i-4}^\Gamma \\
&+ (y_1y_2 - 3)S_{i-5}^\Gamma - (1)S_{i-6}^\Gamma
\end{aligned}$$

Note that since y_i is the character of $\pi_{\omega_i}(\mathcal{A}_2)$ on $c(y_1, y_2)$, the coefficients are themselves characters of virtual representations $\mathcal{A}_2$ evaluated on $c(y_1, y_2)$. For example, since the image of $\pi_{[1,1]}(\mathcal{A}_2)$ in $\mathfrak{R}(\mathcal{A}_2)$ is given by $V_1V_2 - 1$, the coefficient $(y_1y_2 - 3)$ is the character of the virtual representation $\pi_{[1,1]}(\mathcal{A}_2) - 2\pi_{[0,0]}(\mathcal{A}_2)$ evaluated on $c(y_1, y_2)$. Repeating this process for each coefficient, this leads to the following identity on the irreps of $\mathcal{A}_2$:

$$\begin{aligned}
\pi_{[n,n]} = & (\pi_{[1,1]} \ominus 2\pi_{[0,0]}) \otimes (\pi_{[n-1,n-1]} \oplus \pi_{[n-5,n-5]}) \\
& \ominus (\pi_{[3,0]} \oplus \pi_{[0,3]} \ominus \pi_{[1,1]} \oplus 3\pi_{[0,0]}) \otimes (\pi_{[n-2,n-2]} \oplus \pi_{[n-4,n-4]}) \\
& \oplus (\pi_{[2,2]} \ominus \pi_{[3,0]} \ominus \pi_{[0,3]} \oplus 2\pi_{[1,1]} \ominus 3\pi_{[0,0]}) \otimes \pi_{[n-3,n-3]} \\
& \ominus \pi_{[n-6,n-6]}
\end{aligned}$$

This identity is easily checked to hold in general by a similar method to that of

Algorithm 1.1.2. In fact, one further finds that this identity still holds if one replaces the weights $[n, n], \ldots, [n-6, n-6]$ by $[n+I_1, n+I_2], \ldots, [n-6+I_1, n-6+I_2]$ for any weight $[I_1, I_2]$; the proof is analogous. Similar identities can be derived for other Lie Groups by the same diagonal method.

Finally, we note that the above method can also be used to derive recursive $\mathfrak{R}(G)$-linear relationships among the sequences of long and short factors of $\pi_{m[\rho]}(G)$ when G is nonsimply-laced.

Some interesting questions related to the notion of divisible sequences which we would like to answer include:

- Which divisible sequences can arise as divisible subsequences of irreducible characters evaluated on $\mathbb{Z}$-classes in a given G?
- What are necessary and sufficient conditions for a given divisible sequence to arise from some $\mathbb{Z}$-class of a given G?
- Same questions above but for which the $\mathbb{Z}$-class is allowed to be virtual.
- Does every linear recursive divisible sequence arise from evaluating some subsequence of irreducible characters on some (possibly virtual) $\mathbb{Z}$-class of some G? If so, which G can it arise from?

In light of what is known about divisible sequences from [BPP], these questions should not be difficult to answer.

4.3 Factorizations in Plethysms

Besides looking at the factorizations in $\mathfrak{R}(G)$ of irreducible representations, we may also consider factorizations of plethysms of irreducibles. As with calculation of irreducible characters, calculation of plethysm characters is easy since they can be interpreted as irreducible characters of the General Linear group for which the Character Formula applies. However, a cursory glance at some small examples shows that, like factorization of irreps, the general question of factorization in $\mathfrak{R}(G)$ of plethysms of irreps is not at all trivial.

As a first example, all plethysms of the fundamental representation $\pi_{[1]}(\mathcal{A}_1)$ are symmetric powers of this representation; and in fact one has $Sym^k(\pi_{[1]}(\mathcal{A}_1)) = \pi_{[k]}(\mathcal{A}_1)$, so complete factorizations of plethysms of the fundamental representation are already known by our previous work.

On the other hand, if we start with a non-fundamental representation $\pi_{[k]}(\mathcal{A}_1)$ and consider its plethysms, then the resulting representations factor in $\mathfrak{R}(\mathcal{A}_1)$, but the factorization patterns are no longer as easy to described.

In the case of $\mathcal{A}_2$, the plethysms of the fundamental representation $\pi_{[1,0]}(\mathcal{A}_2)$ are all of the form $P_{(i,j)}(\pi_{[1,0]}(\mathcal{A}_2))$ and in fact one has:

$$P_{(i,j)}(\pi_{[1,0]}(\mathcal{A}_2)) = \pi_{[i-j,j]}(\mathcal{A}_2)$$

Thus our work has already covered factorizations of these plethysms. For other irreps of $\mathcal{A}_2$, the plethysms and their factorizations are much more complicated.

As a final example, working in $\mathcal{G}_2$ with the $P_{(n,1)}$-plethysms of $\pi_{[1,0]}(\mathcal{G}_2)$ we have the following results for small n:

Plethysm	Factorization in $\mathfrak{R}(\mathcal{G}_2)$
$P_{(1,1)}(\pi_{[1,0]})$	Irreducible
$P_{(2,1)}(\pi_{[1,0]})$	$(V_1+1)(V_2)$
$P_{(3,1)}(\pi_{[1,0]})$	$(V_1^2-V_1-V_2-1)(V_2)$
$P_{(4,1)}(\pi_{[1,0]})$	$(V_1+1)(V_1^2V_2-2V_2^2-2V_1V_2+V_1^2-V_2-V_1)$
$P_{(5,1)}(\pi_{[1,0]})$	Irreducible

Further calculation indicates that $P_{(n,1)}(\pi_{[1,0]})$ seems to only factor when $\gcd(n,6) > 1$, in such cases the factor $\pi_{[1,0]}+1$ ($n \equiv 0 \mod 2$) or $\pi_{[2,0]}$ ($n \equiv 0 \mod 3$) appears. It is interesting to note that both these common factors are short factors $\widehat{\pi}^S_{m[\rho]}(\mathcal{G}_2)$ with $m = 2, 3$, although there does not appear to be a simple explanation of this phenomenon.

In contrast to the factorizations of the irreps, there are not any easily discernible patterns among the cofactors, whether viewed in $\mathfrak{R}(\mathcal{G}_2)$ or in $\mathfrak{E}(\mathcal{G}_2)$.

On the other hand, the representations $P_{(n,2)}(\pi_{[1,0]})$ do not factor for any small values of n, while the representations $P_{(n,3)}(\pi_{[1,0]})$ again factor in some cases but short factors do not appear in several of these cases.

All these examples deal with relatively simple plethysms of relatively simple representations, and yet exhibit a high degree of complexity, indicating that an extensive study of factorizations of plethysms of irreps of arbitrary G may prove quite fruitful in providing further factorization results. Indeed, we would like to classify what series of factorizations appear in plethysms in general and what the factors of these series look like, as well as determine whether sporadic factorizations appear.

4.4 Factorization after Restriction to a Subgroup

Yet another avenue of study of factorizations involves examining factorization properties of irreps and other representations of G when restricted to a subgroup $H \subset G$. Our work with the Weyl Denominator Formula and its consequences in §3.2.1 through §3.2.3 has already looked at this question in the special case of $\pi_{m[\rho]}(G)$ upon restriction to $\mathbb{T}(G)$; in this case one has $\mathfrak{E}(G) = \mathfrak{R}(\mathbb{T}(G))$. Other examples of restriction from G to a subtorus of $\mathbb{T}(G)$ also afford many simple examples of factorizations which are not yet easily characterized in general.

More interesting and less trivial examples arise from considering other Lie groups and subgroups which are not torii. For example, one may consider $Res_{\mathcal{G}_2}^{\mathcal{F}_4}$ for the standard embedding $\mathcal{G}_2 \hookrightarrow \mathcal{F}_4$. In this case, the restriction of long and short factors of $\mathcal{F}_4$, whether irreducible in $\mathfrak{R}(\mathcal{F}_4)$ or not, always factor in $\mathfrak{R}(\mathcal{G}_2)$:

$$Res_{\mathcal{G}_2}^{\mathcal{F}_4}\left(\widehat{\pi}_m^S(\mathcal{F}_4)\right) = m^3 \cdot \left(\widehat{\pi}_m^S(\mathcal{G}_2)\right)^3$$

$$Res_{\mathcal{G}_2}^{\mathcal{F}_4}\left(\widehat{\pi}_m^L(\mathcal{F}_4)\right) = \widehat{\pi}_m^L(\mathcal{G}_2) \cdot \left(\widehat{\pi}_m^S(\mathcal{G}_2)\right)^3$$

In particular, while the irrep $\widehat{\pi}_2^S(\mathcal{F}_4) = \widehat{\pi}_{[2,2,1,1]}(\mathcal{F}_4)$ does not factor in $\mathfrak{R}(\mathcal{F}_4)$, its restriction to $\mathcal{G}_2$ does factor in $\mathfrak{R}(\mathcal{G}_2)$.

More generally, as noted in §4.3, plethysms naturally arise as the irreducible representations of the general linear groups. Thus, embedding $G \hookrightarrow GL_m(\mathbb{C})$ via a representation $\pi(G)$ of dimension m allows one to consider the plethysms $P_\lambda(\pi(G)$ as the restriction of irreps of $GL_m(\mathbb{C})$. In particular our factorization results for $\mathcal{A}_{m-1}$ extended to the generalized case of $GL_m(\mathbb{C})$ show $P_\lambda(\pi(G)$ will factor for certain partitions λ. However, since restriction from a group to a subgroup corresponds to changes of variables, it is possible that new factors will appear after restriction to a subgroup.

As some examples of the complexity introduced in dealing with restrictions and plethysms, we close by noting that for small n none of the plethysms $P_{(n,1)}(\pi_{[1,0,0]}(\mathcal{B}_3))$ factor. But since $Res_{\mathcal{G}_2}^{\mathcal{B}_3}\left(\pi_{[1,0,0]}(\mathcal{B}_3)\right) = \pi_{[1,0]}(\mathcal{G}_2)$, from our results in §4.3 we know that $Res_{\mathcal{G}_2}^{\mathcal{B}_3}\left(P_{(n,1)}(\pi_{[1,0,0]}(\mathcal{B}_3))\right)$ does factor in many small cases.

As with plethysms, we would like to be able to classify and describe series factorizations appearing in restrictions including how they depend on the pair of groups being considered. Since in general restrictions of irreps are not irreducible, techniques which do not rely as heavily on the Weyl Character Formula will need to be developed to answer these questions.

Chapter 5

Appendix

5.1 MAPLE Routines Used in Calculations

In this section we will outline the MAPLE routines employed in our computations. For purposes of this appendix, all routines will be done for $G = \mathcal{G}_2$ with appropriate comments on the changes for a different Lie group. The necessary packages which must be loaded for these routines are `ListTools`, `LinearAlgebra`, and `Groebner`.

5.1.1 Defining the Weyl Group

We begin by defining the Cartan matrix of $\mathcal{G}_2$ and the matrices of $W(\mathcal{G}_2)$:

$$>\texttt{G2} := \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

$$>\texttt{R} := \texttt{RowDimension(G2)}$$

```
>for k from 1 to R do;
  G_k := IdentityMatrix(R) − Matrix(R, R, (i, j) → piecewise(j = k,
          Row(G2, k)[i], 0));
  end do;
>t1 := M → [seq(seq(M[i, j], j = 1..ColumnDimension(M)),
          i = 1..RowDimension(M))];
>t2 := M → Matrix(R, R, (i, j) → M[R · (i − 1) + j]);
>L_0 := [IdentityMatrix(R)];
>for i from 1 to 6 do;
  assign('K_i', MakeUnique([seq(seq(t1(G_k.L_{i−1}[j]), j = 1..nops(L_{i−1})),
          k = 1..R)]));
  assign('L_i', [seq(t2(K_i[j]), j = 1..nops(K_i))]);
  end do;
  assign('N', [op(L[5]), op(L[6])]);
  unassign('K', 'L');
```

Defining the variable R is not strictly necessary, we include it here only to emphasize which parts of the routines depend on the rank of G versus which ones are independent of the rank. The matrix $\mathtt{G_i}$ corresponds to the action of the simple reflection in the hyperplane perpendicular to the i^{th} fundamental weight of $\mathcal{G}_2$.

The functions `t1` and `t2` are auxillary functions whose definition is the same for any Lie group. `t1` converts an $m \times m$ matrix `M` into a m^2-element list while `t2` is inverse to `t1` and converts an m^2-element list into an $m \times m$ matrix.

The main loop defining the set of matrices in $W(\mathcal{G}_2)$ runs from 1 through the maximal length of an element $g \in W(\mathcal{G}_2)$ where the length of g is the minimal word length of g relative to the set of generators formed by the fundamental reflections. As the length function is well-known for arbitrary Lie groups, the upper bound on this loop can be adjusted accordingly.

Within this loop, the need for the auxillary functions becomes apparent since the `MakeUnique` function in MAPLE can distinguish between lists but not between matrices; hence we first convert the list of matrices corresponding to elements of length i to a list of lists, then remove any repetitions. The following step converts the remaining lists back to matrices, giving the list $\mathtt{L_i}$ consisting of all elements of $W(\mathcal{G}_2)$ of length less than or equal to i. Once this loop is complete, a list `N` is made of all the matrices in the last two lists created; by construction `N` therefore contains matrix representations of all elements of $W(\mathcal{G}_2)$. The final command is optional and is simply a way of freeing up memory for later computations.

5.1.2 The Weyl Character Formula and the ^{LS}-factors

The next step is to implement the Character Formula so that we can effectively work with irreps. MAPLE is quite fast at performing the necessary factoring and divisions, making calculations feasible up through groups of rank 5 and some larger examples.

To implement the Character Formula, we first define the $E_{[I]}(\mathcal{G}_2)$'s that appear, followed by the Character Formula itself:

$$>\mathtt{rho} \ := <\mathtt{seq}(1, \mathtt{i} = 1..\mathtt{R})>;$$

$$>\mathtt{Exp} \ := \ \mathtt{v} \rightarrow \mathtt{product}(\mathtt{X}_{\mathtt{i}}^{\mathtt{v[i]}}, \mathtt{i} = 1..\mathtt{R});$$

$$>\mathtt{E} \ := \ \mathtt{v} \rightarrow \mathtt{add}(\mathtt{Determinant}(\mathtt{N[j]}) \cdot \mathtt{Exp}(\mathtt{N[j]}.(\mathtt{v} + \mathtt{rho})), \mathtt{j} = 1..\mathtt{nops}(\mathtt{N}));$$

$$>\mathtt{Rep} \ := \ \mathtt{v} \rightarrow \mathtt{expand}\left(\mathtt{simplify}\left(\frac{\mathtt{E(v)}}{\mathtt{E}(0 \cdot \mathtt{rho})}\right)\right);$$

The weight `rho` is defined as the vector of size `R` and all entries equal to 1 in accordance with the usual definition of the weight $[\rho]$. The input variable `v` in each of the other three functions is also input as a `R`-dimensional vector $< \mathtt{v}_1, \ldots, \mathtt{v}_{\mathtt{R}} >$. The result of the calling sequence `Rep(I)` is the character $\chi_{[I]}(\mathcal{G}_2)$ expressed as an element of $\mathfrak{E}(\mathcal{G}_2)$. Note that the character is unshifted; the shifted character $\widehat{\chi}_{[I]}(\mathcal{G}_2)$ is given by the calling sequence $\mathtt{Rep}(\mathtt{I} - \rho)$.

In addition to the ordinary definition of the characters given above, we also define the $E^S_{m[\rho]}(\mathcal{G}_2)$ since $\mathcal{G}_2$ is nonsimply-laced. While one could try to obtain a general description of the short roots and then use this description in a simple product structure, we instead define $E^S_{m[\rho]}(\mathcal{G}_2)$ by brute force because it avoids the hassle of finding a general description to use:

$$> \mathtt{ES} \ := \ \mathtt{m} \rightarrow \mathtt{expand}\left(\frac{(\mathtt{X}_1^{\mathtt{m}+1}-1)\cdot(\mathtt{X}_2^{\mathtt{m}+1}-\mathtt{X}_1^{\mathtt{m}+1})\cdot(\mathtt{X}_2^{\mathtt{m}+1}-\mathtt{X}_1^{2\cdot\mathtt{m}+2})}{\mathtt{X}_1^{2\cdot\mathtt{m}+2}\mathtt{X}_2^{\mathtt{m}+1}}\right)$$

The exponent of X_i in the denominator is half the sum of the exponents of X_i in the numerator. Now using Theorem 3.2.4 and the above definition of `ES(n)` we can define the long and short factors in general:

$$>\mathtt{xi} \ := \ \mathtt{u} \rightarrow \mathtt{subs}(\mathtt{X}_1 = \mathtt{z}, \mathtt{X}_2 = \mathtt{X}_1^3, \mathtt{z} = \mathtt{X}_2, \mathtt{u});$$

$$>\mathtt{EL} \ := \ \mathtt{m} \rightarrow \mathtt{xi}(\mathtt{ES}(\mathtt{m}));$$

$$>\mathtt{Short} \ := \ \mathtt{m} \rightarrow \mathtt{expand}\left(\mathtt{simplify}\left(\frac{\mathtt{ES}(\mathtt{m})}{\mathtt{ES}(0)}\right)\right)$$

$$>\mathtt{Long} \ := \ \mathtt{m} \rightarrow \mathtt{expand}\left(\mathtt{simplify}\left(\frac{\mathtt{EL}(\mathtt{m})}{\mathtt{EL}(0)}\right)\right)$$

The function `xi` is exactly the function $\xi_{\mathcal{G}_2}$ as defined in §2.2.1. The need for the dummy variable `z` results from the sequential substitution method employed by MAPLE .

It is not absolutely necessary to define the function $\mathtt{EL}(\mathtt{m})$ in order to define the calling sequence $\mathtt{Long}(\mathtt{m})$ (one could instead define $\mathtt{Long}(\mathtt{m}) \ := \ \mathtt{Xi}(\mathtt{Short}(\mathtt{m}))$), but we will use $\mathtt{EL}(\mathtt{m})$ for later computations so we go ahead and also use it to define $\mathtt{Long}(\mathtt{m})$.

The result of the calling sequence $\mathtt{Short}(\mathtt{m})$ is the character $\chi_m^S(\mathcal{G}_2)$. As with the Character Formula above, note that this is the unshifted character and the shifted character $\widehat{\chi}_m^S(\mathcal{G}_2)$ is given by $\mathtt{Short}(\mathtt{m}-1)$. The analogous comments hold for the calling sequence $\mathtt{Long}(\mathtt{m})$.

5.1.3 Implementing Algorithms 1.1.1 and 1.1.2

Having implemented the Character Formula as well as the Long and Short factors, we are now ready to implement the calculations needed in the main results.

In order to implement Algorithms 1.1.1 and 1.1.2, we first define:

$$>\mathtt{Lead}\ :=\ \mathtt{u} \to \mathtt{LeadingMonomial}(\mathtt{u}, \mathtt{tdeg}(\mathtt{x}_1, \mathtt{x}_2))$$

$$\cdot\mathtt{LeadingCoefficient}(\mathtt{u}, \mathtt{tdeg}(\mathtt{x}_1, \mathtt{x}_2));$$

$$>\mathtt{GLead}\ :=\ \mathtt{u} \to \mathtt{simplify}\Big(\frac{\mathtt{Lead}(\mathtt{subs}(\mathtt{X}_1 = \mathtt{x}_1^6,\ \mathtt{X}_2 = \mathtt{x}_2^{10}, \mathtt{numer}(\mathtt{u})))}{\mathtt{subs}(\mathtt{X}_1 = \mathtt{x}_1^6, \mathtt{X}_2 = \mathtt{x}_2^{10}, \mathtt{denom}(\mathtt{u}))},$$

$$\{\mathtt{x}_1^6 = \mathtt{V}_1, \mathtt{x}_2^{10} = \mathtt{V}_2\}\Big);$$

The function `Lead` simply extracts the highest degree term relative to a monomial weighting scheme. The actual scheme used is defined in `GLead`; here we are giving X_1 a weighting of $\frac{3}{5}$ the weight of X_2. This choice of weighting is not arbitrary; instead, it comes from the notion of the height of a weight as we now explain.

Definition 5.1.1. *The* **height** *function ht of a weight $[I]$ of G is the number of simple weights which are required to express the weight $2[I]$.*

Remark. The simple weights are a different basis for weights than the fundamental weights. The i^{th} simple weight of G is the i^{th} row of the Cartan Matrix $C(G)$ when expressed in terms of the fundamental weights. From its definition it is clear that ht is uniquely defined (since the simple weights are linearly independent,

$2[I]$ has a unique expression as a sum of simple weights) and is a homomorphism from the weight lattice to $\mathbb{Z}$: $ht([I]+[J]) = ht([I]) + ht([J])$.

An exact description of $ht(I)$ requires a little manipulation with $C(G)$ and the definition of ht; one eventually obtains the following expression where $[2I]$ and $[\rho]$ are interpreted as n-vectors:

$$ht([I]) = \left\langle [2I], C(G)^{-1} \cdot [\rho] \right\rangle$$

Remark. This expression as an inner product is not surprising since ht is a linear functional on the weight space of G.

Thus in the case of $\mathcal{G}_2$ for example, one has:

$$\begin{aligned} ht([n_1, n_2]) &= \left\langle [2n_1, 2n_2], \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}^{-1} \cdot [1,1] \right\rangle \\ &= \langle [2n_1, 2n_2], [3,5] \rangle \\ &= 6n_1 + 10n_2 \end{aligned}$$

In our example of $\mathcal{G}_2$, we interpret $ht([n_1, n_2]) = 6n_1 + 10n_2$ as saying that $\exp(\omega_1)$ should have relative weight 6 and $\exp(\omega_2)$ should have relative weight 10 in determining dominance and highest weights, this leads to the given weightings of X_1 and X_2 in the definition of `GLead`.

The key feature of ht that we shall use in implementing Algorithms 1.1.1 and 1.1.2 is that if $[I]$ dominates $[J]$ then $ht([I]) > ht([J])$, but the converse need not be true in general. For Algorithm 1.1.1, to convert a known character `Chi` into an element of $\mathfrak{R}(\mathcal{G}_2)$ one has:

```
>assign('B', Chi);
  assign('L', [ ]);
  while B ≠ 0 do;
  assign('A', GLead(B));
  assign('B', expand(B − subs(V_1 = Rep(< 1, 0 >), V_2 = Rep(< 0, 1 >), A)));
  assign('L', [op(L), [LeadingCoefficient(A, tdeg(V_1, V_2)), degree(A, V_1),
          degree(A, V_2)]]);
  end do;
  assign('L', Matrix(L))
  print(expand(add(L[k, 1] · V_1^L[k,2] · V_2^L[k,3], k = 1..RowDimension(L))));
```

As long as Chi is $W(\mathcal{G}_2)$-symmetric, the while loop will terminate after finitely many steps. The output of this calling sequence is the polynomial in $\mathfrak{R}(\mathcal{G}_2)$ whose character is Chi. At each iteration of the while loop, the height function in GLead picks out a highest weight $[I]$ of multiplicity μ_I from the remaining character B. The rest of the loop is spent recording and subtracting off the character of μ_I times the unique monic monomial in $\mathfrak{R}(\mathcal{G}_2)$ with highest weight $[I]$ thus leaving an expression which either has fewer highest weights whose ht equals $ht([I])$ or has all weights with ht strictly smaller than $ht([I])$.

Algorithm 1.1.2 is implemented similarly, but instead of subtracting off the character of a monomial in $\mathfrak{R}(\mathcal{G}_2)$ one calculates the highest weight $[I]$ of the remaining character at each iteration as before and subtracts off $\texttt{Rep}(<\texttt{I}>)$. As mentioned in the description of Algorithm 1.1.2, by an easy optimization this process can be implemented without first calculating $Rep(< I >)$, thereby saving computation time and memory:

```
>assign('B', Chi · E(0 · rho));
 assign('L', [ ]);
 while B ≠ 0 do;
 assign('A', GLead(B));
 assign('B', expand(B − LeadingCoefficient(A, tdeg(V1, V2))·
          E(< degree(A, V1), degree(A, V2) >)));
 assign('L', [op(L), [LeadingCoefficient(A, tdeg(V1, V2)), degree(A, V1),
          degree(A, V2)]]);
 end do;
 assign('L', Matrix(L))
 print(expand(add(L[k, 1] · pi[[L[k, 2], L[k, 3]]], k = 1..RowDimension(L))));
```

The output is the decomposition of the representation with character $\texttt{Chi}$ into irreducible summands.

Implementing the variant of Algorithm 1.1.2 used to calculate the decompositions of the short factors is slightly more complicated:

$>\texttt{assign}('\texttt{B}', \texttt{expand}(\texttt{ES}(\texttt{m}) \cdot \texttt{E}(0 \cdot \texttt{rho})));$

$\texttt{assign}('\texttt{L}', [\,]);$

$\texttt{while B} \neq 0 \texttt{ do};$

$\texttt{assign}\left('\texttt{A}', \dfrac{\texttt{GLead(B)}}{\texttt{V}_1^3 \cdot \texttt{V}_2}\right);$

$\texttt{assign}('\texttt{L}', [\texttt{op(L)}, [\texttt{LeadingCoefficient}(\texttt{A}, \texttt{tdeg}(\texttt{V}_1, \texttt{V}_2)), \texttt{degree}(\texttt{A}, \texttt{V}_1),$

$\texttt{degree}(\texttt{A}, \texttt{V}_2)]]);$

$\texttt{assign}('\texttt{B}', \texttt{B} - \texttt{LeadingCoefficient}(\texttt{A}, \texttt{tdeg}(\texttt{V}_1, \texttt{V}_2)) \cdot \texttt{expand}(\texttt{ES}(0)\cdot$

$\texttt{E}(< \texttt{degree}(\texttt{A}, \texttt{X}_1), \texttt{degree}(\texttt{A}, \texttt{X}_2) >)));$

$\texttt{end do};$

$\texttt{assign}('\texttt{L}', \texttt{Matrix(L)})$

$\texttt{print}(\texttt{expand}(\texttt{add}(\texttt{L}[\texttt{k}, 1] \cdot \texttt{pi}[[\texttt{L}[\texttt{k}, 2], \texttt{L}[\texttt{k}, 3]]], \texttt{k} = 1..\texttt{RowDimension(L)})));$

The primary differences between this variant and the ordinary implementation of Algorithm 1.1.2 involve the assignments of A and B. In the definition of A the division of GLead(B) by $exp(\epsilon^*[\rho])$ is necessary to adjust for the fact that one is working with short factors. In the case of long factors, the necessary adjustment to GLead(B) instead involves dividing by $\frac{exp(q\cdot[\rho])}{exp(\epsilon^*[\rho]}$ where q is the characteristic of G. Also different from the ordinary implementation above, is that the initial assignment of B involves an extra factor of $ES(i)$ and the looped assignments of B involve an extra factor of ES(0). The end result of this loop is the decomposition of $\pi_m^S(\mathcal{G}_2)$

into irreducible summands.

In the event that one wants to calculate the general decompositions of $\pi_m^S(\mathcal{G}_2)$ and $\pi_m^L(\mathcal{G}_2)$ calculated in Theorem 3.2.7, MAPLE cannot deal with variables in exponents when calculating leading terms, but if one sets $m >> 0$, for example $m = 1000$, then the m-dependent and m-independent portions of the weights are easily discerned. One then extracts the set of m-independent portions of the weights as follows:

$$>\mathtt{assign}('\mathtt{M}', \mathtt{Matrix}(\mathtt{RowDimension}(\mathtt{L}), 3, (\mathtt{i}, \mathtt{j}) \rightarrow \mathtt{piecewise}(\mathtt{j} = 1, \mathtt{L}[\mathtt{i}, 1], \\ \mathtt{j} > 1 \text{ and } \mathtt{L}[\mathtt{i}, \mathtt{j}] > \frac{\mathtt{m}}{2}, \mathtt{L}[\mathtt{i}, \mathtt{j}] - \mathtt{m} + \mathtt{t}, \mathtt{L}[\mathtt{i}, \mathtt{j}])));$$

This call converts the m-dependence of the weights to a variable `t` and stores the resulting variable weights as a new matrix `M`. If the list of general weights is complete then it may be verified for arbitrary `t` by the following command:

$$>\mathtt{simplify}(\mathtt{expand}(\mathtt{ES}(\mathtt{t}) \cdot \mathtt{E}(0 \cdot \mathtt{rho}) - \mathtt{ES}(0) \cdot \mathtt{add}(\mathtt{M}[\mathtt{k}, 1] \cdot \\ \mathtt{E}(< \mathtt{M}[\mathtt{k}, 2], \mathtt{M}[\mathtt{k}, 3] >), \mathtt{k} = 1..\mathtt{RowDimension}(\mathtt{M}))), \mathtt{power}, \mathtt{symbolic});$$

If the output of this calling sequence is 0, then the suspected set of general weights is complete; this is the verification method used in our proofs of Theorems 3.2.7 and 3.2.8.

Bibliography

[AD] J.F. Adams, Lectures on Lie Groups, W.A. Benjamin, Inc. New York-Amsterdam, 1969.

[AD1] J.F. Adams and Z. Mahmud, "Maps Between Classifying Spaces." *Invent. Math* **35** (1976), 1-41.

[AD2] J.F. Adams, "Maps Between Classifying Spaces II." *Invent. Math* **49** (1978), 1-65.

[BD] T. Bröcker and T. tom Dieck, "Representations of Compact Lie Groups", Graduate Texts in Mathematics, 98, Springer-Verlag Inc. New York, 2003.

[BPP] J.P. Bézivin, A. Pethö, A.J. van der Poorten, "A Full Characterization of Divisibility Sequences." Amer. J. of Math. 112 (1990), 985-1001.

[FH] W. Fulton and J. Harris, "Representation Theory. A First Course." Graduate Texts in Mathematics, Readings in Mathematics, 129, Springer-Verlag Inc. New York, 1991.

[Kl] A. Klimyk, "Decomposition of a Direct Product of Irreducible Representations of a Semisimple Lie Algebra into a Direct Sum of Irreducible Representations." *Am. Math. Soc. Translations* **76**, 2 (1968), 63-73.

[LiE] LiE Online Computations,
`http://www-math.univ-poitiers.fr/~maavl/LiE/form.html`

[M11] Waterloo Maple Inc., MAPLE , Version 11.0, Waterloo, Ontario, Canada 2007

[M13] Waterloo Maple Inc., MAPLE , Version 13.0, Waterloo, Ontario, Canada 2009